

The Rosen-Roback Model

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Basic question

in the Rosen-Roback model and more generally in spatial and urban economics

- ▶ How are people (workers) distributed across different locations?
- ▶ For example, Osaka Prefecture has many cities.
 - ▶ Toyonaka, Suita, Ikeda, Ibaraki, Minoh, Osaka, etc...
- ▶ Locations/cities are different in productivity and amenities.
 - ▶ High productivity shifts up production *ceteris paribus*.
 - ▶ High amenities shifts up consumers' utility *ceteris paribus*.
- ▶ Hey, not only people but also capital, firms, and establishments (factories, subsidiaries, and retail stores) are endogenously distributed across space.
 - ▶ This is a central question in urban/spatial economics.
 - ▶ But for now, we focus on labor mobility rather than other production factors' mobility.

Setup

- ▶ The following explanation of the Rosen-Roback model draws on Atsushi Yamagishi and Owen Zider's lecture slides as well as Roback (1982).
- ▶ There are many "small" cities (locations) $i = 1, \dots, n$ and a big outside location (outside option) $n + 1$.
 - ▶ How small? I will elaborate this later.
 - ▶ e.g. all cities in Osaka as "small" cities and the rest of Japan as the big outside location.
 - ▶ Or you can assume a continuum of inside locations $[0, 1]$ and one big outside location.
- ▶ The model is static.
- ▶ Workers are all freely and completely mobile across locations.

Workers' consumption and utility (1)

- ▶ Suppose that an individual has already chosen location i .
- ▶ Assume that her utility in i is

$$u_i = A_i C_i^\alpha (H_i^C)^{1-\alpha}, \quad (1)$$

where

- ▶ A_i is the amenities in location i ,
 - ▶ C_i is the consumption of freely tradeable and homogeneous goods which are the numeraire,
 - ▶ H_i^C is the consumption of land (floor) space,
 - ▶ $\alpha \in (0, 1)$ is the Cobb-Douglas parameter of expenditure shares.
- ▶ A_i is a location-specific parameter.
 - ▶ Assume that each individual supplies a unit of labor inelastically and receives wages as the only source of income.
 - ▶ Then the budget constraint is

$$C_i + r_i H_i^C \leq w_i,$$

where r_i is the rental price (rent) of land.

Workers' consumption and utility (2)

- ▶ Then the constrained maximization problem is

$$\max_{C_i, H_i^C} u_i = A_i C_i^\alpha (H_i^C)^{1-\alpha}$$

subject to

$$C_i + r_i H_i^C \leq w_i.$$

- ▶ Solving this, we have

$$C_i = \alpha w_i, \quad H_i^C = (1 - \alpha) \frac{w_i}{r_i}.$$

- ▶ Substituting these into (1), we have the indirect utility of living in location i

$$V_i = \alpha^\alpha (1 - \alpha)^{1-\alpha} A_i \frac{w_i}{r_i^{1-\alpha}}.$$

- ▶ Assuming that the indirect utility and all relevant terms are positive, and taking logs of both sides,

$$\log V_i = \log w_i - (1 - \alpha) \log r_i + \log A_i + \text{constant}, \quad (2)$$

where constant is $\log(\alpha^\alpha (1 - \alpha)^{1-\alpha})$.

Welfare equalization and spatial equilibrium (1)

- ▶ We have already assumed that workers are freely mobile across locations (including the outside location).
- ▶ Then indirect utilities are equalized across all locations.
 - ▶ Otherwise, workers would move from a location with lower indirectly utility to a location with higher indirect utility.
- ▶ Let \bar{V} be the indirect utility of living in the outside location.
- ▶ Assume that whatever happens in any inside location $i = 1, \dots, n$, \bar{V} does not change.
 - ▶ In this sense, we said we consider "small" cities.
 - ▶ Suppose the entire Tokyo prefecture is an inside location and the rest of Japan is the outside location.
 - ▶ If Tokyo's productivity or amenities double, economic conditions and welfare in the rest of Japan would substantially change.
 - ▶ In this sense, the entire Tokyo prefecture may be too big to be one inside location.
 - ▶ But, I think considering Toyonaka, Ikeda, etc as inside locations is fine.

Welfare equalization and spatial equilibrium (2)

- ▶ In equilibrium for any inside location i ,

$$V_i = \bar{V}. \quad (3)$$

- ▶ The indirect utilities of living in Toyonaka, Ikeda, Ibaraki, etc, and the rest of Japan are the same.
- ▶ In equilibrium, endogenous variables are chosen so that workers are indifferent across locations.
- ▶ To me, this is the most important condition to close spatial models.
- ▶ Substituting (3) into (2), we have

$$\log \bar{V} = \log w_i - (1 - \alpha) \log r_i + \log A_i + \text{constant}.$$

- ▶ We solve this for r_i as

$$\log r_i = \frac{1}{1 - \alpha} \log w_i + \frac{1}{1 - \alpha} \log A_i + \text{constant}', \quad (4)$$

where constant' is constant $- \log \bar{V}$.

Welfare equalization and spatial equilibrium (3)



- ▶ (4) is an indifference curve in the $\log w$ - $\log r$ plane.
- ▶ Workers like high wages and dislike high rent.
- ▶ Therefore, workers are indifferent between a high-rent, high-wage location and a low-rent, low-wage location.
- ▶ This is why the graph shows an upward-sloping line.
- ▶ The original paper by Roback (1982) derives an upward-sloping curve from more general preferences.
 - ▶ Actually we do not need Cobb-Douglas preferences. But, I chose them for the sake of exposition.

Firms and production

- ▶ Firms produce freely tradeable and homogeneous goods with constant-returns-to-scale technologies.
- ▶ In each location, firms have the same production function.
- ▶ We consider a representative firm for each inside location.
- ▶ In location i , the production function is

$$Y_i = Z_i L_i^\beta (H_i^P)^{1-\beta},$$

where

- ▶ Z_i is the productivity in location i (a parameter),
- ▶ L_i is the labor input,
- ▶ H_i^P is the land (or floor space) input,
- ▶ $\beta \in (0, 1)$ is the Cobb-Douglas parameter of cost shares.

Cost minimization

- ▶ Since the wage rate is w_i and the land rent is r_i ,¹ the cost minimization problem in location i is

$$\min_{L_i, H_i^P} w_i L_i + r_i H_i^P$$

subject to

$$Z_i L_i^\beta (H_i^P)^{1-\beta} \geq 1.$$

- ▶ Solving this problem, the optimal labor and land inputs are

$$L_i = \frac{1}{Z_i} \left(\frac{\beta}{1-\beta} \right)^{1-\beta} \left(\frac{r_i}{w_i} \right)^{1-\beta},$$

$$H_i^P = \frac{1}{Z_i} \left(\frac{1-\beta}{\beta} \right)^\beta \left(\frac{w_i}{r_i} \right)^\beta.$$

¹Consumers and firms face the same land/housing market in each inside location.

Unit costs

- ▶ Then, the unit cost function is

$$C(w_i, r_i, Z_i) = \frac{1}{Z_i} \left(\left(\frac{\beta}{1-\beta} \right)^{1-\beta} + \left(\frac{1-\beta}{\beta} \right)^{\beta} \right) w_i^{\beta} r_i^{1-\beta}.$$

- ▶ Assume that firms are perfectly competitive.
- ▶ They earn zero profit.
- ▶ Therefore, the unit cost must be one (the price of homogeneous goods)

$$\frac{1}{Z_i} \left(\left(\frac{\beta}{1-\beta} \right)^{1-\beta} + \left(\frac{1-\beta}{\beta} \right)^{\beta} \right) w_i^{\beta} r_i^{1-\beta} = 1.$$

- ▶ Taking logs of both sides and solving for $\log r_i$,

$$\log r_i = -\frac{\beta}{1-\beta} \log w_i + \frac{1}{1-\beta} \log Z_i + \text{constant}'' , \quad (5)$$

where constant'' is $-\frac{1}{1-\beta} \log \left(\left(\frac{\beta}{1-\beta} \right)^{1-\beta} + \left(\frac{1-\beta}{\beta} \right)^{\beta} \right)$.

Isocost curve



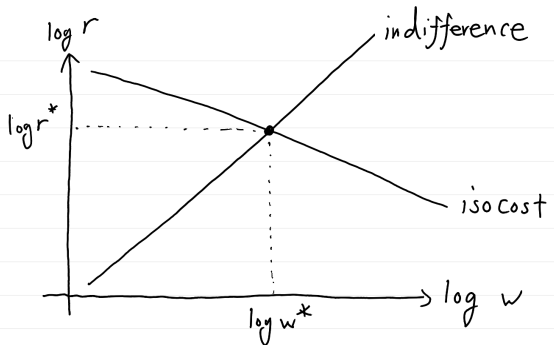
- ▶ (5) is basically the isocost curve in the $\log w$ - $\log r$ plane.
- ▶ Both high wages and high rent makes costs high.
- ▶ This means that unit costs are the same across a high-wage, low-rent location and a low-wage, high-rent location.
- ▶ Therefore, the isocost curve is downward-sloping.
- ▶ Roback (1982) derives a downward-sloping isocost curve from a more general CRS production function.

Caveats

- ▶ In the following, I drop subscript i of w_i and r_i for the sake of exposition.
 - ▶ I actually already did this when I showed graphs.
- ▶ I will discuss equilibrium wages and rent in location i . But, similar arguments hold for any inside location.

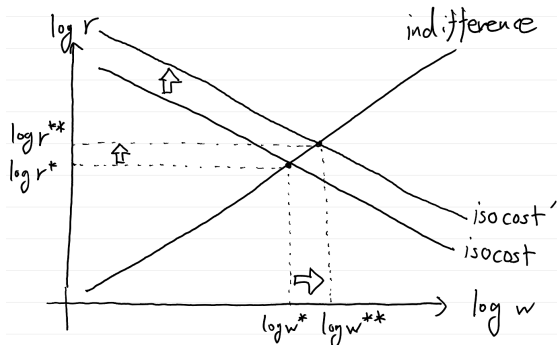
Equilibrium

- ▶ The unique equilibrium is characterized as the only one intersection of the upward-sloping indifference curve and the downward-sloping isocost curve in the $\log w$ - $\log r$ plane for each inside location.



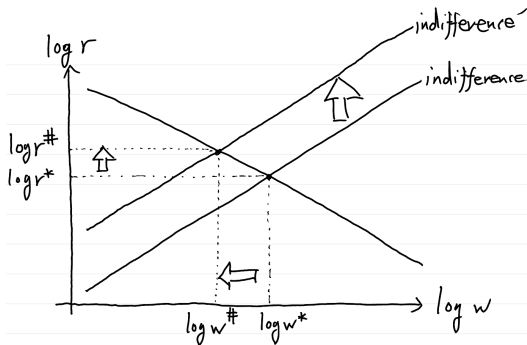
Comparative statics 1: Productivity

- ▶ Suppose that productivity in location i , Z_i , increases.
- ▶ What's the new equilibrium?
- ▶ The isocost curve shifts up in the $\log w$ - $\log r$ plane.
- ▶ The indifference curve stays put.
- ▶ Therefore, both the rent and the wage increase from r^* and w^* to r^{**} and w^{**} .
- ▶ A firm owners says "now productivity is high, so I can pay higher wages and rent."



Comparative statics 2: Amenities

- ▶ Suppose that amenities in location i , A_i , increases.
- ▶ The indifference curve shifts up in the $\log w$ - $\log r$ plane.
- ▶ The isocost curve stays put.
- ▶ Therefore, the rent increases from r^* to $r^\#$.
- ▶ But, the wage decreases from w^* to $w^\#$.
- ▶ "The landscape got better here. I'm ok with lower wages and higher rent."



Comments

- ▶ An advantage and limitation of this model is that each inside location is small relative to the outside location.
- ▶ A change in productivity in a location shifts only the isocost curve of the location.
- ▶ A change in amenities in a location shifts only the indifference curve of the location.
- ▶ And nothing happens in other locations.
- ▶ Wages and rents in each inside location are determined independently of other locations.
- ▶ This is great in the sense that we can neatly tease out the effect of productivity or amenities.
- ▶ But, in the real world, all locations are interrelated.
- ▶ In richer models, changes in productivity or amenities in one location affect other locations in a system of locations.

References I

Roback, J. (1982). Wages, rents, and the quality of life. *Journal of Political Economy*, 90(6):1257–1278.