Fréchet

$P[v_i \epsilon_i > v_j \epsilon_j$ for any $j \neq i]$.

- $\epsilon_1, \cdots, \epsilon_n \overset{iid}{\sim} F(x) = e^{-x^{-\theta}}$.
- $v_1, \cdots, v_n$: positive real numbers.

Compute

$$\Pr[v_i \epsilon_i > v_j \epsilon_j \text{ for any } j \neq i].$$

(e) First we temporarily assume that $\epsilon_i = x$, where x is a deterministic real number.

$$\Pr(v_i \epsilon_i > v_j \epsilon_j \text{ for any } j \neq i \mid \epsilon_i = x).$$

$$= P(v_i x > v_j \epsilon_j \text{ for any } j \neq i).$$

$$= \prod_{j \neq i} P\left(\epsilon_j < \frac{v_i x}{v_j}\right)$$

$$= \prod_{j \neq i} F\left(\frac{v_i x}{v_j}\right).$$

$$= \prod_{j \neq i} \exp\left(-\left(\frac{v_i x}{v_j}\right)^{-\theta}\right).$$

$$= \exp\left(-x^{-\theta} \sum_{j \neq i} \left(\frac{v_i}{v_j}\right)^{-\theta}\right).$$

② Second, we integrate this over x with the weight being

$$f(x) = F'(x) = \theta x^{-(\theta+1)} e^{-x^{-\theta}}.$$

So, we compute

$$\int_0^\infty P(v_i x > v_j \epsilon_j \text{ for any } j \neq i) f(x)\, dx.$$

$$= \int_0^\infty \exp\left(- x^{-\theta} \sum_{j\neq i} \left(\frac{v_i}{v_j}\right)^{-\theta}\right) \cdot \theta x^{-(\theta+1)} e^{-x^{-\theta}}\, dx.$$

$$= \int_0^\infty \theta x^{-(\theta+1)} \exp\left\{-\left(1+\sum_{j\neq i}\left(\frac{v_i}{v_j}\right)^{-\theta}\right)x^{-\theta}\right\} dx.$$

Let $\alpha = 1+\sum_{j\neq i}\left(\frac{v_i}{v_j}\right)^{-\theta}$.

Then,

$$\int_0^\infty \theta x^{-(\theta+1)} \exp(-\alpha x^{-\theta})\, dx.$$

$$= \frac{1}{\alpha} \int_0^\infty \alpha \theta x^{-(\theta+1)} \exp(-\alpha x^{-\theta})\, dx$$

Notice that

$$\exp(-\alpha x^{-\theta})'$$

$$= \alpha \theta x^{-(\theta+1)} \exp(-\alpha x^{-\theta}).$$

Therefore our integral is

$$\frac{1}{\alpha} \left[\exp(-\alpha x^{-\theta}) \right]_0^\infty$$

$$= \frac{1}{\alpha} [1 - 0] = \frac{1}{\alpha}$$

because $\alpha > 0$ and

$$\lim_{x \to 0} \exp\left(-\alpha \frac{1}{x^\theta}\right) \to 0.$$

What we want is

$$\frac{1}{\alpha} = \frac{1}{1 + \sum_{j \neq i} \left(\frac{v_i}{v_j} \right)^{-\theta}}.$$

$$= \frac{1}{1 + v_i^{-\theta} \sum_{j \neq i} v_j^{\theta}}$$

$$= \frac{v_i^{\theta}}{v_i^{\theta} + \sum_{j \neq i} v_j^{\theta}}$$

$$= \frac{v_i^{\theta}}{\sum_{j=1}^{n} v_j^{\theta}}. \qquad //.$$