$$\left[\begin{array}{ll} \max\limits_{s_1,\dots,s_n} & \left(\sum_{i=1}^{n} (v_i s_i)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} . \\ \text{s.t.} & \sum_i s_i = 1 . \end{array}\right.$$

Since

$$\left(\sum_{i=1}^{n} (v_i s_i)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

is just a monotonic transformation of

$$\sum_{i=1}^{n} (v_i s_i)^{\frac{\sigma-1}{\sigma}},$$

the following yields the same maximizer:

$$\left[\begin{array}{ll} \max\limits_{s_1,\dots,s_n} & \sum_{i=1}^{n} (v_i s_i)^{\frac{\sigma-1}{\sigma}} \\ \text{s.t.} & \sum_i s_i = 1 . \end{array}\right.$$

The Lagrangian is

$$L = \sum_{i=1}^{n} (v_i s_i)^{\frac{\sigma-1}{\sigma}}$$
$$+ \lambda \left(1 - \sum_i s_i \right).$$

The first-order conditions are

$$\frac{\partial L}{\partial s_i} = (v_i)^{\frac{\sigma-1}{\sigma}} \cdot \frac{\sigma-1}{\sigma} s_i^{-\frac{1}{\sigma}}$$
$$- \lambda = 0 \text{ for any } i, \quad \cdots \text{ (1)}$$

$$\frac{\partial L}{\partial \lambda} = 1 - \sum_i s_i = 0 \quad \cdots \text{ (2)}$$

(1) implies

$$\frac{(v_i)^{\frac{\sigma-1}{\sigma}} (s_i)^{-\frac{1}{\sigma}}}{(v_1)^{\frac{\sigma-1}{\sigma}} (s_1)^{-\frac{1}{\sigma}}} = 1 \text{ for any } i \neq 1.$$

$$s_i^{\frac{1}{\sigma}} = \left(\frac{v_i}{v_1} \right)^{\frac{\sigma-1}{\sigma}} \cdot s_1^{\frac{1}{\sigma}}.$$

$$s_i^{\frac{1}{\sigma}} = \left(\frac{v_i}{v_1}\right)^{\frac{\sigma-1}{\sigma}} \cdot s_1^{\frac{1}{\sigma}}.$$

$$s_i = \left(\frac{v_i}{v_1}\right)^{\sigma-1} s_1. \quad \cdots \quad ③$$

Substitute ③ into ②:

$$\sum_i s_i = \sum_i \left(\frac{v_i}{v_1}\right)^{\sigma-1} s_1 = 1$$

$$s_1 = \frac{1}{\sum_i \left(\frac{v_i}{v_1}\right)^{\sigma-1}}$$

$$= \frac{1}{(v_1)^{-(\sigma-1)} \sum_i v_i^{\sigma-1}}$$

$$= \frac{v_1^{\sigma-1}}{\sum_i v_i^{\sigma-1}}. \quad \cdots \quad ④$$

Substituting ④ into ③,

$$s_i = \frac{v_i^{\sigma-1}}{\sum_j v_j^{\sigma-1}}. \quad \text{for any } i.$$

The welfare (indirect utility) is

$$\left(\sum_{i=1}^{n} (v_i s_i)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

$$= \left(\sum_{i=1}^{n} \left(v_i \cdot \frac{v_i^{\sigma-1}}{\sum_j v_j^{\sigma-1}} \right)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}.$$

$$= \left[\sum_{i=1}^{n} \left(\frac{v_i^{\sigma}}{\sum_j v_j^{\sigma-1}} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}.$$

$$= \frac{\left[\sum_{i=1}^{n} v_i^{\sigma-1} \right]^{\frac{\sigma}{\sigma-1}}}{\sum_j v_j^{\sigma-1}}$$

$$= \left[\sum_{i=1}^{n} v_i^{\sigma-1} \right]^{\frac{\sigma}{\sigma-1} - 1}$$

$$= \left[\sum_{i=1}^{n} v_i^{\sigma-1} \right]^{\frac{1}{\sigma-1}} \quad //.$$