$$\left[\begin{array}{l} \underset{\{c(\omega)\}_{\omega=0}^{M}}{\text{Max}} \left(\int_0^M c(\omega)^{\frac{\sigma-1}{\sigma}} d\omega\right)^{\frac{\sigma}{\sigma-1}} \quad \cdots \text{①} \\ \text{s.t.} \int_0^M p(\omega)c(\omega)\, d\omega \leq I. \end{array}\right.$$

The maximizer of this problem is the same as the maximizer of

$$\left[\begin{array}{l} \underset{\{c(\omega)\}_{\omega=0}^{M}}{\text{Max}} \int_0^M c(\omega)^{\frac{\sigma-1}{\sigma}} d\omega \quad \cdots \text{②} \\ \text{s.t.} \int_0^M p(\omega)\, c(\omega)\, d\omega \leq I \end{array}\right.$$

because

$$\left(\int_0^M c(\omega)^{\frac{\sigma-1}{\sigma}} d\omega\right)^{\frac{\sigma}{\sigma-1}}$$

is just a monotonic transformation of

$$\int_0^M c(\omega)^{\frac{\sigma-1}{\sigma}} d\omega.$$

Set up the Lagrangian for problem ②:

$$\mathcal{L} = \int_0^M c(\omega)^{\frac{\sigma-1}{\sigma}} d\omega$$

$$+\lambda\left(I - \int_0^M p(\omega)\, c(\omega)\, d\omega\right).$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial c(\omega)} = \frac{\sigma-1}{\sigma} c(\omega)^{-\frac{1}{\sigma}} - \lambda p(\omega) = 0$$

for any $\omega \in [0, M]$, ... ③

$$\frac{\partial \mathcal{L}}{\partial \lambda} = I - \int_0^M p(\omega)\, c(\omega)\, d\omega. \quad \text{... ④}$$

③ implies

$$\left(\frac{c(\omega_1)}{c(\omega_2)}\right)^{-\frac{1}{\sigma}} = \frac{p(\omega_1)}{p(\omega_2)}.$$

$$\left(\frac{c(\omega_1)}{c(\omega_2)}\right)^{-1} = \left(\frac{p(\omega_1)}{p(\omega_2)}\right)^{\sigma}$$

$$c(\omega_2) = \left(\frac{p(\omega_1)}{p(\omega_2)}\right)^{\sigma} c(\omega_1).$$

... ⑤

Substituting ⑤ into ④ (and relabeling ω_2 with ω),

$$I = \int_0^M p(\omega)\left(\left(\frac{p(\omega_1)}{p(\omega)}\right)^{\sigma} c(\omega_1)\right)d\omega$$

$$= c(\omega_1)\, p(\omega_1)^{\sigma} \int_0^M p(\omega)^{1-\sigma} d\omega.$$

$$c(\omega_1) = \frac{p(\omega_1)^{-\sigma}}{\int_0^M p(\omega)^{1-\sigma} d\omega} I.$$

Let $P = \left(\int_0^M p(\omega')^{1-\sigma} d\omega'\right)^{\frac{1}{1-\sigma}}$

be the price index. Then,

$$c(\omega_1) = \frac{p(\omega_1)^{-\sigma}}{P^{1-\sigma}} I$$

$$= \left(\frac{p(\omega_1)}{P}\right)^{-\sigma}\left(\frac{I}{P}\right)$$

Since the choice of ω_1 is arbitrary, for any $\omega \in [0, M]$, we have

$$c(\omega) = \left(\frac{p(\omega)}{P}\right)^{-\sigma} \left(\frac{I}{P}\right) \quad \cdots \textcircled{6}$$

The plugging ⑥ into

$$\left(\int_0^M \left[\left(\frac{p(\omega)}{P}\right)^{-\sigma}\left(\frac{I}{P}\right)\right]^{\frac{\sigma-1}{\sigma}} d\omega\right)^{\frac{\sigma}{\sigma-1}}$$

$$= \left(\int_0^M I^{\frac{\sigma-1}{\sigma}} \cdot P^{(\sigma-1)\frac{\sigma-1}{\sigma}} p(\omega)^{1-\sigma} d\omega\right)^{\frac{\sigma}{\sigma-1}}$$

$$= I \cdot P^{\sigma-1} \cdot \left(\int_0^M p(\omega)^{1-\sigma} d\omega\right)^{\frac{\sigma}{\sigma-1}}$$

$$= I \cdot P^{\sigma-1} \cdot \left(\int_0^M p(\omega)^{1-\sigma} d\omega\right)^{-\frac{\sigma}{1-\sigma}}$$

$$= I \cdot P^{\sigma-1} \cdot P^{-\sigma} = \frac{I}{P}.$$

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