Solving CES Utility Maximization with Continuous Varieties Using the Gateaux Derivative

1 The Problem

Consider a representative consumer who consumes a continuum of differentiated varieties indexed by

$$i \in [0, M].$$

Let $q(i) \geq 0$ denote consumption of variety i, and let $p(i) > 0$ denote the price of variety i. The consumer has income $I > 0$.

The CES utility aggregator is

$$U[q] = \left(\int_0^M q(i)^\rho \, di \right)^{1/\rho},$$

where

$$0 < \rho < 1.$$

The consumer solves

$$\max_{q \geq 0} \left(\int_0^M q(i)^\rho \, di \right)^{1/\rho}$$

subject to the budget constraint

$$\int_0^M p(i) q(i) \, di = I.$$

Since the function $x \mapsto x^{1/\rho}$ is strictly increasing, this problem is equivalent to

$$\max_{q \geq 0} \int_0^M q(i)^\rho \, di$$

subject to

$$\int_0^M p(i) q(i) \, di = I.$$

Thus, we study the problem

$$\max_{q \geq 0} F[q] \equiv \int_0^M q(i)^\rho \, di$$

subject to

$$B[q] \equiv \int_0^M p(i)q(i)\,di = I.$$

2 Function Space and Assumptions

To keep the argument simple, assume that $M < \infty$, $0 < \rho < 1$, and p is measurable and bounded away from zero and infinity:

$$0 < \underline{p} \leq p(i) \leq \overline{p} < \infty$$

for almost every $i \in [0, M]$.

We choose q from the set of nonnegative measurable functions satisfying

$$\int_0^M p(i)q(i)\,di < \infty.$$

Under these assumptions, the problem has an interior solution with

$$q(i) > 0$$

for almost every i. The first-order condition can therefore be derived using two-sided perturbations around the optimum.

3 The Gateaux Derivative

The choice variable is not a finite-dimensional vector. It is the whole function

$$q(\cdot).$$

Therefore, the derivative we need is a derivative of a functional.

Let

$$F[q] = \int_0^M q(i)^\rho\,di.$$

For a perturbation direction h, define

$$q_\varepsilon(i) = q(i) + \varepsilon h(i).$$

Here h is an admissible perturbation direction. For example, one may take $h \in L^\infty([0, M])$, with ε sufficiently small so that

$$q(i) + \varepsilon h(i) \geq 0.$$

The Gateaux derivative of F at q in direction h is defined by

$$DF[q](h) = \frac{d}{d\varepsilon} F[q + \varepsilon h]\bigg|_{\varepsilon=0}.$$

Since

$$F[q + \varepsilon h] = \int_0^M (q(i) + \varepsilon h(i))^\rho \, di,$$

we have

$$DF[q](h) = \frac{d}{d\varepsilon} \int_0^M (q(i) + \varepsilon h(i))^\rho \, di\bigg|_{\varepsilon=0}.$$

Under the regularity assumptions above, differentiation under the integral is justified. Therefore,

$$DF[q](h) = \int_0^M \rho q(i)^{\rho-1} h(i) \, di.$$

Thus, the Gateaux derivative is the linear map

$$h \mapsto \int_0^M \rho q(i)^{\rho-1} h(i) \, di.$$

Equivalently, the variational derivative is

$$\frac{\delta F}{\delta q(i)} = \rho q(i)^{\rho-1}.$$

4 The Lagrangian Functional

The Lagrangian functional is

$$\mathcal{L}[q, \mu] = \int_0^M q(i)^\rho \, di - \mu \left(\int_0^M p(i) q(i) \, di - I \right),$$

where μ is the Lagrange multiplier.

The Gateaux derivative of the Lagrangian with respect to q in direction h is

$$D_q \mathcal{L}[q, \mu](h) = \frac{d}{d\varepsilon} \mathcal{L}[q + \varepsilon h, \mu]\bigg|_{\varepsilon=0}.$$

We compute

$$\mathcal{L}[q + \varepsilon h, \mu] = \int_0^M (q(i) + \varepsilon h(i))^\rho \, di - \mu \left(\int_0^M p(i)(q(i) + \varepsilon h(i)) \, di - I \right).$$

Differentiating with respect to ε at $\varepsilon = 0$, we obtain

$$D_q \mathcal{L}[q, \mu](h) = \int_0^M \rho q(i)^{\rho-1} h(i) \, di - \mu \int_0^M p(i) h(i) \, di.$$

Therefore,

$$D_q\mathcal{L}[q,\mu](h) = \int_0^M \left[\rho q(i)^{\rho-1} - \mu p(i)\right] h(i)\, di.$$

5 The First-Order Condition

At an interior optimum q^*, the Gateaux derivative of the Lagrangian must be zero in every admissible direction h. Hence

$$D_q\mathcal{L}[q^*,\mu](h) = 0$$

for every admissible h.

That is,

$$\int_0^M \left[\rho q^*(i)^{\rho-1} - \mu p(i)\right] h(i)\, di = 0$$

for every admissible h.

Since this must hold for every perturbation direction h, it follows that

$$\rho q^*(i)^{\rho-1} - \mu p(i) = 0$$

for almost every $i \in [0, M]$.

Thus, the first-order condition is

$$\rho q^*(i)^{\rho-1} = \mu p(i)$$

for almost every i.

This is the rigorous version of the informal expression

$$\frac{\partial \mathcal{L}}{\partial q(i)} = 0.$$

6 Solving for Demand

From the first-order condition,

$$\rho q^*(i)^{\rho-1} = \mu p(i).$$

Rearranging,

$$q^*(i)^{\rho-1} = \frac{\mu}{\rho} p(i).$$

Since $\rho - 1 < 0$, we get

$$q^*(i) = \left(\frac{\mu}{\rho} p(i)\right)^{1/(\rho-1)}.$$

Define

$$\sigma \equiv \frac{1}{1-\rho}.$$

Since $0 < \rho < 1$, we have $\sigma > 1$. Notice that

$$\frac{1}{\rho - 1} = -\frac{1}{1-\rho} = -\sigma.$$

Therefore,

$$q^*(i) = Ap(i)^{-\sigma},$$

where

$$A \equiv \left(\frac{\mu}{\rho}\right)^{-\sigma}.$$

Now impose the budget constraint:

$$\int_0^M p(i)q^*(i)\,di = I.$$

Substituting $q^*(i) = Ap(i)^{-\sigma}$, we obtain

$$\int_0^M p(i)Ap(i)^{-\sigma}\,di = I.$$

Hence

$$A\int_0^M p(i)^{1-\sigma}\,di = I.$$

Therefore,

$$A = \frac{I}{\int_0^M p(j)^{1-\sigma}\,dj}.$$

Thus the Marshallian demand for variety i is

$$q^*(i) = \frac{Ip(i)^{-\sigma}}{\int_0^M p(j)^{1-\sigma}\,dj}.$$

Equivalently, since

$$1 - \sigma = -\frac{\rho}{1-\rho},$$

we may write

$$q^*(i) = \frac{Ip(i)^{-1/(1-\rho)}}{\int_0^M p(j)^{-\rho/(1-\rho)}\,dj}.$$

7 Price Index Representation

Define the CES price index

$$P = \left(\int_0^M p(i)^{1-\sigma}\,di\right)^{1/(1-\sigma)}.$$

Then

$$P^{1-\sigma} = \int_0^M p(i)^{1-\sigma}\, di.$$

Therefore the demand function can be written as

$$q^*(i) = I \frac{p(i)^{-\sigma}}{P^{1-\sigma}}.$$

Equivalently,

$$q^*(i) = \frac{I}{P}\left(\frac{p(i)}{P}\right)^{-\sigma}.$$

This is the familiar CES demand function.

8 Indirect Utility

Using the CES price index, indirect utility is

$$V(p, I) = \frac{I}{P}.$$

To see this, recall that the optimal demand is

$$q^*(i) = \frac{I}{P}\left(\frac{p(i)}{P}\right)^{-\sigma}.$$

Then

$$U[q^*] = \left(\int_0^M q^*(i)^\rho\, di\right)^{1/\rho}.$$

The standard CES duality result gives

$$U[q^*] = \frac{I}{P}.$$

Thus, the cost of attaining one unit of CES utility is P, and the maximum utility attainable with income I is I/P.

9 Why the First-Order Condition Is Sufficient

The objective function

$$F[q] = \int_0^M q(i)^\rho\, di$$

is strictly concave in q, because x^ρ is strictly concave for $0 < \rho < 1$.

The feasible set

$$\left\{ q \geq 0 : \int_0^M p(i)q(i)\, di = I \right\}$$

is convex, because it is defined by a linear equality constraint and a nonnegativity constraint.

Therefore, the optimization problem is a concave maximization problem. Since the candidate solution satisfies the first-order condition and the budget constraint, it is the global maximizer.

Moreover, strict concavity implies uniqueness up to equality almost everywhere.

10 Connection with the Informal Derivation

In many economics texts, the Lagrangian is written as

$$\mathcal{L} = \int_0^M q(i)^\rho \, di - \mu \left(\int_0^M p(i)q(i) \, di - I \right),$$

and the first-order condition is written informally as

$$\frac{\partial \mathcal{L}}{\partial q(i)} = 0.$$

This gives

$$\rho q(i)^{\rho-1} - \mu p(i) = 0.$$

Strictly speaking, this is not an ordinary partial derivative, because $q(i)$ is part of a function $q(\cdot)$, not a finite-dimensional vector. The rigorous interpretation is:

$$D_q \mathcal{L}[q, \mu](h) = 0$$

for every admissible perturbation direction h.

That is,

$$\int_0^M \left[\rho q(i)^{\rho-1} - \mu p(i) \right] h(i) \, di = 0$$

for every admissible h. Since h is arbitrary, the term in brackets must be zero almost everywhere:

$$\rho q(i)^{\rho-1} = \mu p(i).$$

Thus, the informal derivative

$$\frac{\partial \mathcal{L}}{\partial q(i)}$$

is shorthand for the variational derivative

$$\frac{\delta \mathcal{L}}{\delta q(i)}.$$

11 Summary

The continuous-variety CES utility maximization problem is

$$\max_{q\geq 0}\left(\int_0^M q(i)^\rho\,di\right)^{1/\rho}$$

subject to

$$\int_0^M p(i)q(i)\,di = I.$$

Since the outer power is monotone, this is equivalent to maximizing

$$\int_0^M q(i)^\rho\,di.$$

Using the Gateaux derivative, the first-order condition is

$$\rho q(i)^{\rho-1} = \mu p(i)$$

for almost every i.

Solving this condition and imposing the budget constraint gives

$$q(i) = \frac{Ip(i)^{-\sigma}}{\int_0^M p(j)^{1-\sigma}\,dj},$$

where

$$\sigma = \frac{1}{1-\rho}.$$

Equivalently, using the CES price index

$$P = \left(\int_0^M p(i)^{1-\sigma}\,di\right)^{1/(1-\sigma)},$$

demand can be written as

$$q(i) = \frac{I}{P}\left(\frac{p(i)}{P}\right)^{-\sigma}.$$