Constant Elasticity of Substitution (CES) Functions

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Review

Individuals' discrete choices ⇒ aggregate shares

- Last time, we have discussed the individual-level discrete choice problem with the Fréchet and Gumbel distributions and its aggregate consequence.
- There each infinitesimal individual makes a location choice.
- Under our assumptions, the choice probabilities are interpreted as the population shares.
- Today, we derive similar share and welfare formulae under a different setup.

Setup

- There are n locations.
- There is a single household with a unit mass of members.
- The household head allocates members across locations.
- If the household head assigns share s_i of members to location i, the household's collective utility is

$$\left(\sum_{i=1}^{n} (v_i s_i)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}, \tag{1}$$

  where
  - parameter $v_i > 0$ captures the marginal contribution of a member in location i to utility,
  - parameter $\sigma > 1$ is the elasticity of substitution across locations.
- This type of utility functions is called the constant-elasticity-of-substitution (CES) utility function.
- The resource constraint is $\sum_{i=1}^{n} s_i = 1$.

The maximization problem (1)

- Therefore, the household solves

$$\max_{s_1,\cdots,s_n} \left(\sum_{i=1}^{n} (v_i s_i)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

subject to

$$\sum_{i=1}^{n} s_i = 1.$$

The maximization problem (2)

- Since
$$\left(\sum_{i=1}^{n}(v_i s_i)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$
is just a monotonic transformation of
$$\sum_{i=1}^{n}(v_i s_i)^{\frac{\sigma-1}{\sigma}},$$
the following problem yields the same maximizer (but not the same value function):
$$\max_{s_1,\cdots,s_n} \sum_{i=1}^{n}(v_i s_i)^{\frac{\sigma-1}{\sigma}}$$
subject to
$$\sum_{i=1}^{n} s_i = 1.$$

Applying the Lagrangian multiplier method (1)

- The Lagrangian is

$$L = \sum_{i=1}^{n} (v_i s_i)^{\frac{\sigma-1}{\sigma}} + \lambda \left(1 - \sum_i s_i \right).$$

- The first-order conditions are

$$\frac{\partial L}{\partial s_i} = v_i^{\frac{\sigma-1}{\sigma}} \frac{\sigma - 1}{\sigma} s_i^{-\frac{1}{\sigma}} - \lambda = 0 \tag{2}$$

for any i, and

$$\frac{\partial L}{\partial \lambda} = 1 - \sum_i s_i = 0. \tag{3}$$

- (2) implies

$$\frac{v_i^{\frac{\sigma-1}{\sigma}} s_i^{-\frac{1}{\sigma}}}{v_1^{\frac{\sigma-1}{\sigma}} s_1^{-\frac{1}{\sigma}}} = 1$$

for any i.

Applying the Lagrangian multiplier method (2)

- Rearranging the last equation in the previous page, we get

$$s_i = \left(\frac{v_i}{v_1}\right)^{\sigma-1} s_1 \tag{4}$$

  for any i.
- Substituting (4) into (3),

$$\sum_i s_i = \sum_i \left(\frac{v_i}{v_1}\right)^{\sigma-1} s_1 = 1.$$

- Rearranging this, we have

$$s_1 = \frac{v_1^{\sigma-1}}{\sum_i v_i^{\sigma-1}}. \tag{5}$$

- Substituting (5) into (4),

$$s_i = \frac{v_i^{\sigma-1}}{\sum_j v_j^{\sigma-1}} \tag{6}$$

  for any location i.

Welfare

Indirect utility

- Substituting the shares (6) into the objective function (1), we have the indirect utility

$$
\begin{aligned}
&\left(\sum_{i=1}^{n}\left(v_i \frac{v_i^{\sigma-1}}{\sum_{j=1}^{n} v_j^{\sigma-1}}\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \\
=&\left[\sum_i \left(\frac{v_i^{\sigma}}{\sum_j v_j^{\sigma-1}}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \\
=&\frac{\left(\sum_i v_i^{\sigma-1}\right)^{\frac{\sigma}{\sigma-1}}}{\sum_j v_j^{\sigma-1}} \\
=&\left(\sum_i v_i^{\sigma-1}\right)^{\frac{\sigma}{\sigma-1}-1} = \left(\sum_i v_i^{\sigma-1}\right)^{\frac{1}{\sigma-1}} .
\end{aligned}
$$

Formulae from Fréchet and CES

	Fréchet	CES
population share	$\frac{v_i^\theta}{\sum_j v_j^\theta}$	$\frac{v_i^{\sigma-1}}{\sum_j v_j^{\sigma-1}}$
welfare	$\Gamma\left(1-\frac{1}{\theta}\right)\left(\sum_{i=1}^n v_i^\theta\right)^{1/\theta}$	$\left(\sum_i v_i^{\sigma-1}\right)^{\frac{1}{\sigma-1}}$

- Ignoring a constant from the gamma function, if $\theta = \sigma - 1$, the share and welfare formulae from the individual discrete choices with Fréchet preference shocks and the CES representative household are the same.
- See Anderson et al. (1987), Verboven (1996), and Berger et al. (2022) for more details on the equivalence between discrete choices with extreme value distributions and CES representative households.

What is σ?

- Equation (6) implies that the population ratio between locations i and j is

$$\frac{s_i}{s_j} = \left(\frac{v_i}{v_j}\right)^{\sigma-1}.$$

- Hence, σ determines how sensitively the population ratio responds to changes in relative "attractiveness" (or "appeal") across locations.
- If σ is high, many people move from j to i as v_i/v_j increases:
  - Locations are highly substitutable.
- If σ is low, few people move from j to i as v_i/v_j increases:
  - Locations are weakly substitutable.
- Similar interpretations apply for the shape parameter of the Fréchet distribution θ.

CES of continuous varieties (1)

- The CES utility function is applied to varieties which have different prices, too.
- Suppose the following utility maximization problem.
- Mass M of varieties are available for a consumer.
  - Varieties are indexed by $\omega \in [0, M]$.
- Her nominal income (or wealth) is $I > 0$.
- She is a price taker facing a continuum of prices $\{p(\omega)\}_{\omega=0}^{M}$.
- Then the maximization problem is

$$\max_{\{c(\omega)\}_{\omega=0}^{M}} \left(\int_0^M c(\omega)^{\frac{\sigma-1}{\sigma}} d\omega \right)^{\frac{\sigma}{\sigma-1}}$$

subject to

$$\int_0^M p(\omega)c(\omega)d\omega \leq I.$$

CES of continuous varieties (2)

- The demand for variety ω is

$$c(\omega) = \Bigg(\underbrace{\frac{p(\omega)}{P}}_{\text{relative price}} \Bigg)^{-\sigma} \underbrace{\frac{I}{P}}_{\text{real income}},$$

  where P is the price index

$$P = \left(\int_0^M p(\omega')^{1-\sigma} d\omega' \right)^{\frac{1}{1-\sigma}}.$$

- The indirect utility (welfare) is

$$I/P.$$

  - The real income is the welfare.

CES of continuous varieties (3)

- See 3_2_CES_varieties.pdf for derivations.
- You can show that P is the expenditure function (the value function of the expenditure minimization problem) for the utility level one.
- This type of utility and demand functions are very widely used in international and spatial economics.
  - To name a few: Krugman (1979), Melitz (2003), and Anderson and van Wincoop (2003).

References I

Anderson, J. E. and van Wincoop, E. (2003). Gravity with gravitas: A solution to the border puzzle. *American Economic Review*, 93(1):170–192.

Anderson, S. P., De Palma, A., and Thisse, J.-F. (1987). The ces is a discrete choice model? *Economics Letters*, 24(2):139–140.

Berger, D., Herkenhoff, K., and Mongey, S. (2022). Labor market power. *American Economic Review*, 112(4):1147–93.

Krugman, P. R. (1979). Increasing returns, monopolistic competition, and international trade. *Journal of International Economics*, 9(4):469–479.

Melitz, M. J. (2003). The impact of trade on intra-industry reallocations and aggregate industry productivity. *Econometrica*, 71(6):1695–1725.

Verboven, F. (1996). The nested logit model and representative consumer theory. *Economics Letters*, 50(1):57–63.