Goods trade, factor mobility and welfare
Redding (2016)

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November 6, 2025

Introduction

- In the last class, we studied Hsieh and Moretti (2019).
- In that model, individuals consume both freely tradeable goods and housing.
- However, delivering goods across locations involves transportation costs.
- Redding (2016) incorporates such costly trade into a quantitative spatial model.
- In this framework, locations trade goods with each other following the structure of Eaton and Kortum (2002).
  - Eaton and Kortum (2002) is one of the most influential models in international trade.
- In my (subjective) view, Redding (2016) is the most standard static quantitative spatial model.

Model: setup

- The discrete set of locations is N.
- Locations are different in land supply, productivity, amenities and their geographical locations relative to one another.
- Bilateral trade costs for goods take the iceberg form.
  - Samuelson made this concept.
- d_{ni} units of a good must be shipped from location i for one unit to arrive in location n, where $d_{ni} > 1$ for $n \neq i$.
- $d_{nn} = 1$ for any location n.

Consumer preferences (1)

- Worker ω's utility in location i is

$$U_n(\omega) = b_n(\omega) \left(\frac{C_n(\omega)}{\alpha} \right)^{\alpha} \left(\frac{H_{Un}(\omega)}{1-\alpha} \right)^{1-\alpha}, \qquad (1)$$

  where
  - $C_n(\omega)$ is ω's goods consumption in n,
  - H_{Un} is ω's residential land use in n,
  - $b_n(\omega)$ is ω's idiosyncratic amenity shock for n,
  - $\alpha \in (0,1)$ is the parameter of expenditure shares.
- Dropping worker index ω, the goods consumption index is defined over consumption of a fixed continuum of goods $j \in [0,1]$:

$$C_n = \left[\int_0^1 c_n(j)^{\rho} dj \right]^{\frac{1}{\rho}},$$

  where the CES parameter (ρ) determines the elasticity of substitution between goods ($\sigma = 1/(1-\rho)$).

Consumer preferences (2)

- The corresponding dual price index for goods consumption (P_n) is

$$P_n = \left[\int_0^1 p_n(j)^{1-\sigma} dj\right]^{\frac{1}{1-\sigma}}, \ \sigma = \frac{1}{1-\rho}.$$

- The idiosyncratic amenity shock ($b_n(\omega)$) capture the idea that workers have heterogeneous preferences for living in each location.
- These amenity shocks are drawn independently across locations and workers from a Fréchet distribution

$$G_n(b) = e^{-B_n b^{-\epsilon}},$$

  where
  - the scale parameter B_n determines average amenities for location n,
  - the shape parameter ϵ controls the dispersion of amenities across workers for each location.
- Workers inelastically supply one unit of labor.

Production (1)

- The production side of this model follows Eaton and Kortum (2002).
- Each location draws an idiosyncratic productivity $z(j)$ for each good j.
- Productivity is independently drawn across goods and locations from a Fréchet distribution

$$F_i(z) = e^{-A_i z^{-\theta}},$$

  where
  - the scale parameter A_i determines average productivity for location i,
  - the shape parameter θ controls the dispersion of productivity across goods.

Production (2)

- Each good is produced with labor under conditions of perfect competition according to a linear technology.
  - For simplicity, land is not used for production.
- The cost to a consumer in location n of purchasing one unit of good j from location i is

$$p_{ni}(j) = \frac{d_{ni} w_i}{z_i(j)}, \tag{2}$$

where w_i denotes the wage in location i.

Expenditure shares and price indices (1)

- The representative consumer in location n sources each good from the lowest-cost supplier to that location.
- Using equilibrium prices (2) and the properties of the Fréchet distribution following Eaton and Kortum (2002), the share of expenditure of location n on goods produced by location i is

$$\pi_{ni} = \frac{A_i (d_{ni} w_i)^{-\theta}}{\sum_{s \in N} A_s (d_{ns} w_s)^{-\theta}}.$$

- The elasticity of trade with respect to trade costs (trade elasticity) is governed by the Fréchet shape parameter θ.
- See my slides for detailed derivations.

Expenditure shares and price indices (2)

- Using the domestic trade share (π_{nn}) and noting that $d_{nn} = 1$, the consumption goods price index is written as

$$P_n^{-\theta} = \gamma^{-\theta} \left[\sum_{i \in N} A_i (d_{ni} w_i)^{-\theta} \right] = \frac{\gamma^{-\theta} A_n w_n^{-\theta}}{\pi_{nn}}, \qquad (3)$$

  where $\gamma = \left[\Gamma \left(\frac{\theta - (\sigma - 1)}{\theta} \right) \right]^{\frac{1}{1-\sigma}}$ and $\Gamma(\cdot)$ denotes the Gamma function.
- Assume $\theta > \sigma - 1$ to ensure a finite value for the price index.

Residential choices and income (1)

- Given the specification of consumer preferences (1), the indirect utility function is

$$U_n(\omega) = \frac{b_n(\omega) v_n}{P_n^{\alpha} r_n^{1-\alpha}},$$

  where
  - r_n is the land rent for location n,
  - v_n is income in location n.
- v_n is different from w_n because income from land rents is redistributed to the residents of each location.
  - Some people (including my former supervisor) don't like this assumption.
  - This is because if one moves from one location to another, she will suddenly get rent revenues from the destination.
  - But this assumption leads to very neat equilibrium conditions.

Residential choices and income (2)

- Since indirect utility is a linear function of the amenity draw, it too has a Fréchet distribution:

$$G_n(U) = e^{-\Psi_n U^{-\epsilon}},\ \Psi_n = B_n \left(\frac{v_n}{P_n^{\alpha} r_n^{1-\alpha}} \right)^{\epsilon}.$$

- Each worker chooses the location that offers her the highest utility after taking into account her idiosyncratic preferences.
- The probability that a worker chooses to live in location $n \in N$ is

$$\frac{L_n}{\bar{L}} = \frac{B_n(v_n/(P_n^{\alpha} r_n^{1-\alpha}))^{\epsilon}}{\sum_{k \in N} B_k(v_k/(P_k^{\alpha} r_k^{1-\alpha}))^{\epsilon}}. \quad (4)$$

- The elasticity of population with respect to real income is governed by the Fréchet shape parameter ϵ.

Residential choices and income (3)

- Expected utility for a worker across locations is

$$\bar{U} = \delta \left[\sum_{k \in N} B_k (v_k / (P_k^{\alpha} r_k^{1-\alpha}))^{\epsilon} \right]^{\frac{1}{\epsilon}},$$

  where $\delta = \Gamma((\epsilon - 1)/\epsilon)$.
- We require $\epsilon > 1$ to ensure a finite value of $\bar{U}$.

Implications of the Fréchet distribution

- Expected utility conditional on living in location n is the same across all locations.
- And this equals expected utility for the economy as a whole.
- On the one hand, more attractive location characteristics raise the utility of a worker with a given idiosyncratic utility draw.
- On the other hand, more attractive location characteristics attract workers with lower idiosyncratic utility draws.
- These two effects exactly offset one another.

Residential choices and income (4)

- Expenditure on land in each location is redistributed lump sum to the workers in that location.
- Therefore total income in each location (v_n) equals labor income plus expenditure on residential land

$$v_n L_n = w_n L_n + (1-\alpha) v_n L_n = \frac{w_n L_n}{\alpha}. \tag{5}$$

- The trade balance condition is

$$w_i L_i = \sum_{n \in L} \pi_{ni} w_n L_n.$$

- The land market clearing is

$$r_n H_n = (1-\alpha) v_n L_n.$$

- Assume that land supply H_n is exogenously given.
- Solving this for r_n, we have

$$r_n = \frac{(1-\alpha) v_n L_n}{H_n} = \frac{1-\alpha}{\alpha} \frac{w_n L_n}{H_n}. \tag{6}$$

General equilibrium

- An equilibrium of the model is a tuple of $\{L_n\}_{n\in N}$, $\{\pi_{ni}\}_{(n,i)\in N\times N}$, $\{w_n\}_{n\in N}$ that satisfies

$$w_i L_i = \sum_{n\in L} \pi_{ni} w_n L_n, \tag{7}$$

$$\pi_{ni} = \frac{A_i (d_{ni} w_i)^{-\theta}}{\sum_{k\in N} A_k (d_{nk} w_k)^{-\theta}}, \tag{8}$$

$$\frac{L_n}{\bar{L}} = \frac{B_n \left(\frac{A_n}{\pi_{nn}}\right)^{\frac{\alpha\epsilon}{\theta}} \left(\frac{L_n}{H_n}\right)^{-\epsilon(1-\alpha)}}{\sum_{k\in N} B_k \left(\frac{A_k}{\pi_{kk}}\right)^{\frac{\alpha\epsilon}{\theta}} \left(\frac{L_k}{H_k}\right)^{-\epsilon(1-\alpha)}}.$$

- Substituting (3) and (6) into (4), we get the third equation.

Quasi-symmetric trade costs

- To establish uniqueness of the equilibrium, we impose an additional assumption.
- Transport costs (d_{ni}) are partitioned into
  - an importer component (D_n),
  - an exporter component (D_i),
  - a symmetric bilateral component ($D_{ni} = D_{in}$),

$$d_{ni} = \begin{cases} 1 & \text{if } n = i, \\ D_n D_i D_{ni} & \text{if } n \neq i, \end{cases}$$

  where $D_n > 1, D_i > 1$, and $D_{ni} = D_{in} > 1$.
- I don't know why Redding calls this quasi-symmetric rather than symmetric.

Existence and uniqueness (1)

- Then the system of equilibrium conditions reduces to

$$
\begin{aligned}
& L_n^{\tilde{\theta}\gamma_1} A_n^{-\tilde{\theta}} B_n^{-\frac{\tilde{\theta}(1+\theta)}{\alpha\epsilon}} H_n^{-\frac{\tilde{\theta}(1+\theta)(1-\alpha)}{\alpha}} \\
=& \bar{W}^{-\theta}\gamma^{-\theta}\left[\sum_{k\in N} d_{nk}^{-\theta} A_k^{\frac{\tilde{\theta}(1+\theta)}{\theta}} B_k^{\frac{\tilde{\theta}\theta}{\alpha\epsilon}} H_k^{\frac{\tilde{\theta}\theta(1-\alpha)}{\alpha}} \left(L_k^{\tilde{\theta}\gamma_1}\right)^{\frac{\gamma_2}{\gamma_1}}\right], \qquad (9)
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{W} &= \left[\alpha^{\epsilon}\left(\frac{1-\alpha}{\alpha}\right)^{\epsilon(1-\alpha)} (\bar{U}/\delta)^{\epsilon}\bar{L}^{-1}\right]^{1/\alpha\epsilon}, \\
\tilde{\theta} &= \frac{\theta}{1+2\theta}, \\
\gamma_1 &= 1+(1+\theta)\left(\frac{1}{\alpha\epsilon}+\frac{1-\alpha}{\alpha}\right), \\
\gamma_2 &= 1-\theta\left(\frac{1}{\alpha\epsilon}+\frac{1-\alpha}{\alpha}\right) < \gamma_1.
\end{aligned}
$$

Existence and uniqueness (2)

- Equilibrium expected utility ($\bar{U}$) is implicitly determined by the requirement $\sum_{n \in N} L_n = \bar{L}$.

Existence and uniqueness (3)

Proposition 1

Given

- the land area, productivity, and amenity parameters $\{H_n, A_n, B_n\}$,
- the quasi-symmetric bilateral trade frictions $\{d_{ni}\}$

for all location $n, i \in N$, there exists unique equilibrium populations (L_n^*), wages (w_n^*) and trade shares (π_{ni}^*).

Existence and uniqueness (4)

- Redding (2016) does not detailedly prove this.
- Rather, he cites Allen and Arkolakis (2014) (a spatial economics paper) and Fujimoto and Krause (1985) (a math paper).
- Temporarily forget $\bar{W}$ and $\bar{U}$.
- The main equilibrium condition (9) would be linear if $\gamma_1 = \gamma_2$.
  - Then, all we have to do is to scrutinize the properties of a matrix.
- But now we have $\gamma_1 > \gamma_2$, so (9) is non-linear equations.
- Nevertheless, under some conditions, a unique fixed point is guaranteed.
- Especially, if $0 < \gamma_2 < \gamma_1$, the right-hand side is a linear combination of concave functions of $\{L_k^{\tilde{\theta}\gamma_1}\}$.
- In such a case, Fujimoto and Krause (1985) guarantee a unique fixed point.

An algorithm to compute equilibria

- Iterating (9) should be one way to compute an equilibrium.
- A more common way to compute an equilibrium is the following.
- Let ϵ_0 and ϵ_1 be small positive numbers such that $\epsilon_1 < \epsilon_0$.

1. Guess $(L_n)_{n \in N}$ such that $\sum_n L_n = \bar{L}$.
   1.1 Guess $(w_n)_{n \in N}$.
   1.2 Given such $(w_n)_{n \in N}$, compute $\pi_{ni} = \frac{A_i (d_{ni} w_i)^{-\theta}}{\sum_k A_k (d_{nk} w_k)^{-\theta}}$.
   1.3 Compute $w_i^{\text{new}} = \frac{1}{L_i} \sum_n \pi_{ni} w_n L_n$. If $\|(w^{\text{new}} - w)/w\| < \epsilon_1$, we're done. Otherwise, go back to 1.1 with our new guess w^{new}.
   1.4 Using the converged w, compute $P_n = \gamma \left[\sum_i A_i (d_{ni} w_i)^{-\theta}\right]^{-1/\theta}$ and $r_n = \frac{1-\alpha}{\alpha} \frac{w_n L_n}{H_n}$.
2. Compute $L_n^{\text{new}} = \frac{B_n \left(\frac{w_n}{P_n^{\alpha} r_n^{1-\alpha}}\right)^{\epsilon}}{\sum_k B_k \left(\frac{w_k}{P_k^{\alpha} r_k^{1-\alpha}}\right)^{\epsilon}} \bar{L}$. If $\|(L^{\text{new}} - L)/L\| < \epsilon_0$, we're done. Otherwise, Go back to 1 with the new guess L^{new}.

Model inversion

Proposition 3

- As in Hsieh and Moretti (2019), we can invert the model to recover model parameters given elasticities.
- This time the inversion is a bit more complicated because of trade costs.

Given

- the model parameters $\{\alpha, \theta, \epsilon\}$,
- a parameterization of bilateral trade costs $\{d_{ni}\}$,
- data on populations, wages, and land supplies $\{L_n, w_n, H_n\}$,

there exist unique values of amenities (B_n) and productivities (A_n) that are consistent with the data...

- ... up to a normalization that corresponds to a choice of units in which to measure amenities and productivities.

Model inversion: algorithm (1)

- Redding does not explicitly show algorithms to implement model inversion.
- Here I explain algorithms I used for model inversion.
- Suppose that all parameters but productivity $\{A_n\}_{n \in N}$ and amenities $\{B_n\}_{n \in N}$ are known.
- Substituting (8) into (7) yields

$$
\begin{aligned}
w_i L_i &= \sum_n \left\{ \frac{A_i (d_{ni} w_i)^{-\theta}}{\sum_k A_k (d_{nk} w_k)^{-\theta}} \right\} w_n L_n \\
&= A_i W_i^{-\theta} \sum_n \left\{ \frac{d_{ni}^{-\theta}}{\sum_k A_k (d_{nk} w_k)^{-\theta}} \right\} w_k L_n .
\end{aligned}
$$

- Solving this for A_i, we get

$$
A_i = f(A) = \frac{w_i^{\theta+1} L_i}{\sum_n \frac{d_{ni}^{-\theta} w_n L_n}{\sum_k A_k (d_{nk} w_k)^{-\theta}}}, \tag{10}
$$

where function f is defined by the right-most term, and $A = (A_k)_{k \in N}$.

Model inversion: algorithm (2)

1. Guess $A^{(k)}$.
   - The very initial guess is $A^{(0)}$. The superscript (k) is the index for the number of iterations.
2. Then update the productivity vector using (10)

$$A^{(k+1)} = f(A^{(k)}).$$

3. If $A^{(k)}$ and $A^{(k+1)}$ are close enough, that is, if $\|(A^{(k+1)} - A^{(k)})/A^{(k)}\| < \epsilon$ for some norm $\|\cdot\|$ and a predetermined small positive number ϵ, we stop here. Otherwise, using $A^{(k+1)}$ as our new guess and go back to 2.

- Note that function f is homogeneous of degree one, so A is identified up to normalization.
  - This means that we cannot the absolute level of productivity.
  - But, we can measure how good productivity is in location 1 relative to location 2.

Model inversion: algorithm (3)

- Now we got the productivity vector A.
- Then we can recover price indices using (3)

$$P_n = \gamma \left[\sum_i A_i (d_{ni} w_i)^{-\theta} \right]^{-1/\theta}.$$

- (5) implies $v_n = \frac{w_n}{\alpha}$.
- Substituting this into (4) yields

$$\frac{L_n}{\bar{L}} = \frac{B_n \left(\frac{w_n/\alpha}{P_n^\alpha r_n^{1-\alpha}} \right)^\epsilon}{\sum_k B_k \left(\frac{w_k/\alpha}{P_k^\alpha r_k^{1-\alpha}} \right)^\epsilon} = \frac{B_n \left(\frac{w_n}{P_n^\alpha r_n^{1-\alpha}} \right)^\epsilon}{\sum_k B_k \left(\frac{w_k}{P_k^\alpha r_k^{1-\alpha}} \right)^\epsilon}. \tag{11}$$

Model inversion: algorithm (4)

- Define l_n and ζ_k by

$$l_n = \frac{L_n}{\bar{L}},\ \zeta_n = \left(\frac{w_n}{P_n^{\alpha} r_n^{1-\alpha}}\right)^{\epsilon}.$$

- Then (11) can be written as

$$B_n = \frac{l_n}{\zeta_n} \sum_k \zeta_k B_k. \tag{12}$$

- With abuse of notations, let N be the number of locations (it was the set, not the cardinality initially).
- Then (12) is rewritten as

$$\underbrace{\begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} (l_1/\zeta_1)\zeta_1 & (l_1/\zeta_1)\zeta_2 & \cdots & (l_1/\zeta_1)\zeta_N \\ (l_2/\zeta_2)\zeta_1 & (l_2/\zeta_2)\zeta_2 & \cdots & (l_2/\zeta_2)\zeta_N \\ \vdots & \vdots & \ddots & \vdots \\ (l_N/\zeta_N)\zeta_1 & (l_N/\zeta_N)\zeta_2 & \cdots & (l_N/\zeta_N)\zeta_N \end{bmatrix}}_{\Delta} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix}. \tag{13}$$

Model inversion: algorithm (5)

- As in the last line in the previous page, we define vector $B = (B_k)_{k=1}^N$ and matrix Δ whose (i, j) component is $\Delta_{i,j} = (l_i/\zeta_i)\zeta_j$.
- Then (13) is simplified to

$$B = \Delta B.$$

- This means that matrix Δ has an eigenvalue one and its associated eigenvector is B.
- Notice that Δ's components are all positive.
- Then the Perron–Frobenius theorem implies
  1. there is a unique eigenvalue of largest magnitude (absolute value),
  2. the corresponding eigenvector can be chosen to have all strictly positive components.

Model inversion: algorithm (6)

- Therefore, the PF theorem guarantees that there is a scalar a and a strictle positive vector $\tilde{B}$ such that

$$a\tilde{B} = \Delta\tilde{B}.$$

- But, for any scalar $b > 0$,

$$ab\tilde{B} = \Delta(b\tilde{B}).$$

- So ab is also an eigenvalue and $b\tilde{B}$ is its corresponding eigenvector.
- This means that we can identify B up to normalization (similar to the unit dependence problem in the last lecture).
- Here it has an economic sense.
- We cannot measure the absolute level of utility or amenities.
- But based on revealed preferences, we can measure location 1's amenities relative to location 2's.

References I

Allen, T. and Arkolakis, C. (2014). Trade and the topography of the spatial economy *. *The Quarterly Journal of Economics*, 129(3):1085–1140.

Eaton, J. and Kortum, S. (2002). Technology, geography, and trade. *Econometrica*, 70(5):1741–1779.

Fujimoto, T. and Krause, U. (1985). Strong ergodicity for strictly increasing nonlinear operators. *Linear Algebra and its Applications*, 71:101–112.

Hsieh, C.-T. and Moretti, E. (2019). Housing constraints and spatial misallocation. *American Economic Journal: Macroeconomics*, 11(2):1–39.

Redding, S. J. (2016). Goods trade, factor mobility and welfare. *Journal of International Economics*, 101:148–167.