# The Melitz-Chaney Model

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# Motivation

- In the Krugman model, all firms have the same productivity, and all firms export.
- But, in reality, only a small fraction of firms export, and exporting firms are larger and more productive.
  - Large employment (the number of employees or total work hours),
  - Large sales,
  - High value-added per worker.
  - See, for example, Bernard, Jensen, Redding, and Schott (2007), p.125.
- Therefore, we introduce firm heterogeneity.
  - So that only productive firms export in the model.

# Setup

- In the following, we discuss a simple version of the Melitz-Chaney model following Allen and Arkolakis's lecture notes.
- The set of countries is $S$.
- Each country $i \in S$ is populated by an exogenous measure $L_i$ of workers/consumers.
- Each worker supplies a unit of labor inelastically.
- Labor is the only factor of production.

# Varieties and firms (1)

- Each firm produces a different variety.
- The set of varieties produced in country $i$ is denoted by $\Omega_i$.
  - $\Omega_i$ is an endogenous object.
- The set of varieties produced in the world is $\Omega = \bigcup_{i \in S} \Omega_i$.
  - But, in equilibrium, a subset of $\Omega$ is not consumed by consumers in a country if the subset is not exported to the country.
- There is a mass $M_i$ of firms in country $i$.
- Firms in country $i$ must incur a fixed cost $f_{ij}$ to export to destination $j$.

# Varieties and firms (2)

- Firms are heterogeneous.
- Each firm in country $i$ draws productivity $\varphi$ from a cumulative distribution function $G_i(\varphi)$.
  - It costs a productivity-$\varphi$ firm $1/\varphi$ units of labor to produce one unit of its variety.
  - Henceforth, we say "firm $\varphi$" because firms that have the same productivity behave in the same way.
- All firms are subject to iceberg trade costs $\{\tau_{ij}\}_{i,j \in S}$.

# Preferences and budget constraints

- The utility of consumers in $j$ is

$$U_j = \left( \sum_{i \in S} \int_{\Omega_{ij}} \left( q_{ij}(\omega) \right)^{\frac{\sigma - 1}{\sigma}} d\omega \right)^{\frac{\sigma}{\sigma - 1}}.$$

  - $\Omega_{ij}$: the set of varieties produced in $i$ and available in $j$.
  - $q_{ij}(\omega)$: the demand of variety $\omega$ shipped from $i$ to $j$.
  - $\sigma$: the elasticity of substitution.
- The budget constraint is

$$\sum_{i \in S} \int_{\Omega_{ij}} p_{ij}(\omega) q_{ij}(\omega) d\omega = Y_j.$$

  - $p_{ij}(\omega)$: the price of $\omega$ from $j$ that consumers in $i$ face.
  - $Y_j$: the total expenditure in $j$.

# Optimal demand

- The demand for $\omega$ produced in $i$ by consumers in $j$ is

$$q_{ij}(\omega) = \left(\frac{p_{ij}(\omega)}{P_j}\right)^{-\sigma} \frac{Y_j}{P_j}.$$

  where

$$P_j = \left(\sum_{i \in S} \int_{\Omega_{ij}} p_{ij}(\omega)^{1-\sigma} d\omega\right)^{\frac{1}{1-\sigma}}.$$

- The amount spend on variety $\omega$ is

$$x_{ij}(\omega) = p_{ij}(\omega) q_{ij}(\omega) = p_{ij}(\omega)^{1-\sigma} Y_j P_j^{\sigma-1}.$$

- Integrating this across all varieties, the aggregate trade value from $i$ to $j$ is

$$X_{ij} = \int_{\Omega_{ij}} x_{ij}(\omega) d\omega = Y_j P_j^{\sigma-1} \int_{\Omega_{ij}} p_{ij}(\omega)^{1-\sigma} d\omega. \tag{1}$$

# Optimal prices

- Firm $\varphi$ in $i$ sets the optimal prices (across various destinations) to maximize its profits
$$\max_{\{p_{ij}(\varphi)\}_{j\in S}} \sum_{j\in S} \left( p_{ij}(\varphi) q_{ij}(\varphi) - \frac{w_i}{\varphi} \tau_{ij} q_{ij}(\varphi) - f_{ij} \right)$$
such that
$$q_{ij}(\varphi) = p_{ij}(\varphi)^{-\sigma} Y_j P_j^{\sigma-1}.$$
- Substituting the demand functions into the maximand yields
$$\max_{\{p_{ij}(\varphi)\}_{j\in S}} \sum_{j\in S} \left( p_{ij}(\varphi)^{1-\sigma} Y_j P_j^{\sigma-1} - \frac{w_i}{\varphi} \tau_{ij} p_{ij}(\varphi)^{-\sigma} Y_j P_j^{\sigma-1} - f_{ij} \right).$$
- The first-order consition characterizes the optimal price that firm $\varphi$ from $i$ sets in $j$
$$p_{ij}(\varphi) = \frac{\sigma}{\sigma-1} \frac{w_i}{\varphi} \tau_{ij}.$$

# Trade values and operating profits

- Firm $\varphi$'s trade value from $i$ to $j$ conditional on it serving $j$ is

$$x_{ij}(\varphi) = p_{ij}(\varphi)q_{ij}(\varphi) = \left(\frac{\sigma}{\sigma-1}\frac{w_i}{\varphi}\tau_{ij}\right)^{1-\sigma} Y_j P_j^{\sigma-1}. \tag{2}$$

- The operating profits that firm $\varphi$ from $i$ earning in $j$ is

$$\begin{aligned}
\pi_{ij}(\varphi) &= \left(p_{ij}(\varphi) - \frac{w_i}{\varphi}\tau_{ij}\right) q_{ij}(\varphi) \\
&= \left(\frac{\sigma}{\sigma-1}\frac{w_i}{\varphi}\tau_{ij} - \frac{w_i}{\varphi}\tau_{ij}\right)\left(\frac{\sigma}{\sigma-1}\frac{w_i}{\varphi}\tau_{ij}\right)^{-\sigma} Y_j P_j^{\sigma-1} \\
&= \frac{1}{\sigma}\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma}\left(\frac{w_i}{\varphi}\tau_{ij}\right)^{1-\sigma} Y_j P_j^{\sigma-1} \\
&= \frac{1}{\sigma}x_{ij}(\varphi).
\end{aligned} \tag{3}$$

# Price indices and trade values

- Let $h_{ij}(\varphi)$ be the probability density function of productivity of firms from country $i$ that sells to $j$.
- Then we have

$$\begin{aligned} & \int_{\Omega_{ij}} p_{ij}(\varphi)^{1-\sigma} d\omega \\ = & \int_0^\infty M_{ij} \left( \frac{\sigma}{\sigma-1} \frac{w_i}{\varphi} \tau_{ij} \right)^{1-\sigma} h_{ij}(\varphi) d\varphi \\ = & M_{ij} \left( \frac{\sigma}{\sigma-1} w_i \tau_{ij} \right)^{1-\sigma} (\tilde{\varphi}_{ij})^{\sigma-1}, \end{aligned} \tag{4}$$

  where $\tilde{\varphi}_{ij} = \left( \int_0^\infty \varphi^{\sigma-1} h_{ij}(\varphi) d\varphi \right)^{1/(\sigma-1)}$ is what Melitz called the "average" productivity of firms that sell from $i$ to $j$.
- Then we can rewrite the aggregate trade flow, (1), as

$$X_{ij} = \left( \frac{\sigma}{\sigma-1} \right)^{1-\sigma} \tau_{ij}^{1-\sigma} w_{1-\sigma} M_{ij} (\tilde{\varphi}_{ij})^{\sigma-1} Y_j P_j^{\sigma-1}.$$

# Selection into exporting and cutoff productivity

- Firm $\varphi$ in country $i$ exports to $j$ if and only if its operating profits exceeds the fixed cost to export there
$$\pi_{ij}(\varphi) \geq f_{ij}.$$
- Using (2) and (3), we rewrite this condition as
$$\frac{1}{\sigma}\left(\frac{\sigma}{\sigma-1}\frac{w_i}{\varphi}\tau_{ij}\right)^{1-\sigma} Y_j P_j^{\sigma-1} \geq f_{ij}.$$
That is, productivity $\varphi$ exceeds the cutoff productivity $\varphi_{ij}^*$
$$\varphi \geq \varphi_{ij}^* = \left(\frac{\sigma f_{ij}(\frac{\sigma}{\sigma-1} w_i \tau_{ij})^{\sigma-1}}{Y_j P_j^{\sigma-1}}\right)^{\frac{1}{\sigma-1}}. \tag{5}$$

# What's $h_{ij}$?

- For $\varphi < \varphi_{ij}^*$, $h_{ij}(\varphi) = 0$.
- For $\varphi \geq \varphi_{ij}^*$,

$$h_{ij}(\varphi) = \frac{g_i(\varphi)}{\int_{\varphi_{ij}^*}^{\infty} g_i(\varphi) d\varphi} = \frac{g_i(\varphi)}{1 - G_i(\varphi_{ij}^*)}.$$

# Cutoffs pin down masses of exporters

- The "average" productivity of firms selling from $i$ to $j$ is

$$\tilde{\varphi}_{ij} = \left( \frac{1}{1 - G_i(\varphi_{ij}^*)} \int_{\varphi_{ij}^*}^{\infty} \varphi^{\sigma-1} dG_i(\varphi) \right)^{\frac{1}{\sigma-1}}.$$

- The mass of firms selling from $i$ to $j$ is

$$M_{ij} = (1 - G_i(\varphi_{ij}^*)) M_i.$$

- The aggregate trade value from $i$ to $j$ is

$$X_{ij} = \left( \frac{\sigma}{\sigma - 1} \right)^{1-\sigma} \tau_{ij}^{1-\sigma} w_i^{1-\sigma} M_i \left( \int_{\varphi_{ij}^*}^{\infty} \varphi^{\sigma-1} dG_i(\varphi) \right) Y_j P_j^{\sigma-1}. \tag{6}$$

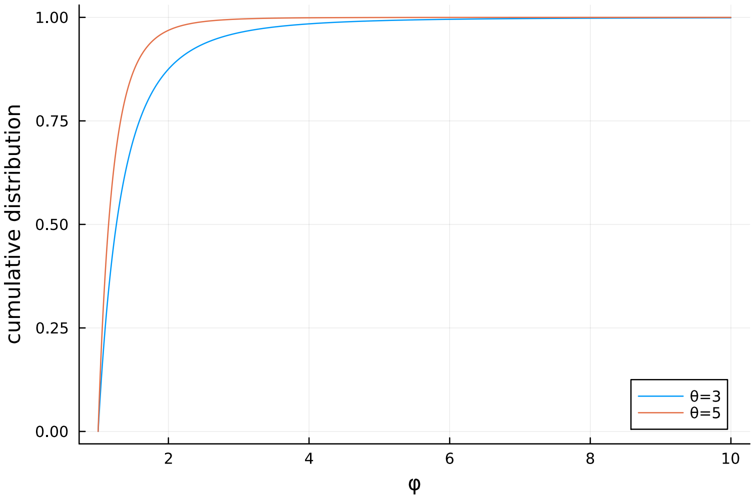
# The Pareto distribution (Chaney)

- So far we have not specified $G_i(\cdot)$.
- For simplicity, suppose that productivity $\varphi$ is no less than 1 in any country $\varphi \in [1, \infty)$.
- Assume that productivity of firms in $i$ follows the Pareto distribution with shape parameter $\theta_i$

$$G_i(\varphi) = 1 - \varphi^{-\theta_i}. \tag{7}$$

- Assume $\theta_i > \sigma - 1$ so that trade flows are finite.

# Examples of the Pareto distribution




See pareto.jl for the code to produce this figure.

# The Pareto distributed productivity leads to the gravity equation (1)

- A part of the average productivity is

$$
\begin{aligned}
\int_{\varphi_{ij}^*}^{\infty} \varphi^{\sigma-1} dG_i(\varphi) &= \int_{\varphi_{ij}^*}^{\infty} \varphi^{\sigma-1} \left( \frac{d(1-\varphi^{-\theta_i})}{d\varphi} \right) d\varphi \\
&= \theta_i \int_{\varphi_{ij}^*}^{\infty} \varphi^{\sigma-\theta_i-2} d\varphi \\
&= \frac{\theta_i}{\theta_i+1-\sigma} (\varphi_{ij}^*)^{\sigma-\theta_i-1} \\
&= \frac{\theta_i}{\theta_i+1-\sigma} \left( \frac{\sigma f_{ij} (\frac{\sigma}{\sigma-1} w_i \tau_{ij})^{\sigma-1}}{Y_j P_j^{\sigma-1}} \right)^{\frac{\sigma-\theta_i-1}{\sigma-1}},
\end{aligned}
\tag{8}
$$

where the last equality follows from the cutoff (5).

# The Pareto distributed productivity leads to the gravity equation (2)

- (6) and (8) yield

$$
\begin{aligned}
X_{ij} &= \left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \tau_{ij}^{1-\sigma} w_i^{1-\sigma} M_i \left( \frac{\theta_i}{\theta_i+1+\sigma} \left( \frac{\sigma f_{ij} (\frac{\sigma}{\sigma-1} w_i \tau_{ij})^{\sigma-1}}{Y_j P_j^{\sigma-1}} \right)^{\frac{\sigma-\theta_i-1}{\sigma-1}} \right) Y_j P_j^{\sigma-1} \\
&= C_{1,i} (\tau_{ij} w_i)^{-\theta_i} f_{ij}^{\frac{\sigma-\theta_i-1}{\sigma-1}} M_i (Y_j P_j^{\sigma-1})^{\frac{\theta_i}{\sigma-1}},
\end{aligned}
\tag{9}
$$

where $C_{1,i} = \sigma^{\frac{\sigma-\theta_i-1}{\sigma-1}} \left(\frac{\sigma}{\sigma-1}\right)^{-\theta_i} \left(\frac{\theta_i}{\theta_i+1-\sigma}\right)$.

# Free entry (1)

- Firms have to incur an entry cost $f_i^e$ *before* they learn their productivity.
- The free entry condition is that the expected profits are equal to the entry cost

$$f_i^e = E_\varphi \left[ \sum_{j \in S} \max\{\pi_{ij}(\varphi) - f_{ij}, 0\} \right].$$

- Then we can rewrite the equation above as

$$\begin{aligned} f_i^e &= \int_1^\infty \sum_{j \in S} \max\{\pi_{ij}(\varphi) - f_{ij}, 0\} dG_i(\varphi) \\ &= \sum_{j \in S} \int_{\varphi_{ij}^*}^\infty (\pi_{ij}(\varphi) - f_{ij}) dG_i(\varphi). \end{aligned} \tag{10}$$

# Free entry (2)

- With the Pareto distribution (7), we can rewrite (10) as

$$f_i^e = \sum_{j \in S} \frac{\sigma - 1}{\theta_i + 1 - \sigma} \left( \frac{\sigma}{\sigma - 1} w_i \tau_{ij} \right)^{-\theta_i} \sigma^{-\frac{\theta_i}{\sigma-1}} f_{ij}^{\frac{\sigma-\theta_i-1}{\sigma-1}} Y_j^{\frac{\theta_i}{\sigma-1}} P_j^{\theta_i}.$$

- Country $i$'s only source of income is wages, so we have

$$Y_j = w_j L_j. \tag{11}$$

- Therefore we have

$$f_i^e = \sum_{j \in S} \frac{\sigma - 1}{\theta_i + 1 - \sigma} \left( \frac{\sigma}{\sigma - 1} w_i \tau_{ij} \right)^{-\theta_i} \sigma^{-\frac{\theta_i}{\sigma-1}} f_{ij}^{\frac{\sigma-\theta_i-1}{\sigma-1}} (w_j L_j)^{\frac{\theta_i}{\sigma-1}} P_j^{\theta_i}. \tag{12}$$

# Rewriting price indices (1)

- The average productivity is

$$\begin{aligned}\tilde{\varphi}_{ij}^{\sigma-1} &= \frac{1}{1-G_i(\varphi_{ij}^*)}\int_{\varphi_{ij}^*}^{\infty}\varphi^{\sigma-1}dG_i(\varphi) \\ &= \frac{1}{(\varphi_{ij}^*)^{-\theta_i}}\frac{\theta_i}{\theta_i+1-\sigma}(\varphi_{ij}^*)^{\sigma-\theta_i-1} \\ &= \frac{\theta_i}{\theta_i+1-\sigma}\frac{\sigma f_{ij}(\frac{\sigma}{\sigma-1}w_i\tau_{ij})^{\sigma-1}}{Y_jP_j^{\sigma-1}}.\end{aligned} \tag{13}$$

- (4) and (13) yield

$$P_j^{1-\sigma} = \sum_{i\in S} M_{ij}\left(\frac{\sigma}{\sigma-1}w_i\tau_{ij}\right)^{1-\sigma}\frac{\theta_i}{\theta_i+1-\sigma}\frac{\sigma f_{ij}(\frac{\sigma}{\sigma-1}w_i\tau_{ij})^{\sigma-1}}{Y_jP_j^{\sigma-1}}.$$

# Rewriting price indices (2)

- The last equation is rewritten as

$$
\begin{aligned}
Y_j &= \sum_{i \in S} M_{ij} \frac{\theta_i}{\theta_i + 1 - \sigma} \sigma f_{ij} \\
&= \sum_{i \in S} (1 - G(\varphi_{ij}^*)) M_i \frac{\theta_i}{\theta_i + 1 - \sigma} \sigma f_{ij} \\
&= \sum_{i \in S} (\varphi_{ij}^*)^{-\theta_i} M_i \frac{\theta_i}{\theta_i + 1 - \sigma} \sigma f_{ij} \\
&= \sum_{i \in S} \left( \frac{\sigma f_{ij} \left( \frac{\sigma}{\sigma - 1} w_i \tau_{ij} \right)^{\sigma - 1}}{Y_j P_j^{\sigma - 1}} \right)^{-\frac{\theta_j}{\sigma - 1}} M_i \frac{\theta_i}{\theta_i + 1 - \sigma} \sigma f_{ij} \\
&= \sum_{i \in S} (\sigma f_{ij})^{-\frac{\theta_j}{\sigma - 1}} \left( \frac{\sigma}{\sigma - 1} w_i \tau_{ij} \right)^{-\theta_j} Y_j^{\frac{\theta_j}{\sigma - 1}} P_j^{\theta_j} M_i \frac{\theta_i}{\theta_i + 1 - \sigma} \sigma f_{ij}.
\end{aligned}
\tag{14}
$$

# Rewriting price indices (3)

- Solving (14) for $P_j^{-\theta_j}$, we have

$$P_j^{-\theta_j} = \sum_{i \in S} \frac{\theta_i}{\theta_i + 1 - \sigma} M_i (\sigma f_{ij})^{\frac{-\theta_j + \sigma - 1}{\sigma - 1}} \left( \frac{\sigma}{\sigma - 1} w_i \tau_{ij} \right)^{-\theta_j} (w_j L_j)^{\frac{\theta_j - \sigma + 1}{\sigma - 1}}. \tag{15}$$

# Trade balance

- Assume that the income is equal to the expenditure (trade balance)

$$Y_j = \sum_{i \in S} X_{ij}.$$

- This, (9), and (11) yield

$$w_j L_j = \sum_{i \in S} C_{1,i} (\tau_{ij} w_i)^{-\theta_i} f_{ij}^{\frac{\sigma - \theta_i - 1}{\sigma - 1}} M_i (w_j L_j P_j^{\sigma - 1})^{\frac{\theta_i}{\sigma - 1}}. \tag{16}$$

# Equilibrium system

- Now we end up with a system of $3N$ equations (12), (15), (16) with $3N$ unknowns $\{w_i\}_{i \in S}$, $\{M_i\}_{i \in S}$, and $\{P_i\}_{i \in S}$.
- This characterizes an equilibrium.