Production Networks

Motoaki Takahashi

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Background

- Goods and services are produced through supply chains.
  - Coal is used in steel production; steel is used in truck production; trucks are used in postal services (sector-to-sector).
  - Hyundai sells battery packs to Volkswagen; Volkswagen sells busses to a bus company (firm-to-firm).
- How are such production networks formed?
- Related questions:[1]
  - How does a shock in a particular sector/firm transmit through production networks?
  - Anticipating geopolitical risks, how should a country form supply chain networks with foreign countries?

[1] These are not addressed in the model we will focus on, though.

Literature (1)

- There is a massive and still growing literature on production networks.
- Classic
  - Hulten (1978)
- Sector-to-sector
  - Baqaee and Farhi (2019a), Baqaee and Farhi (2024), Baqaee (2018), Baqaee and Farhi (2019b)
  - Acemoglu and Azar (2020), Acemoglu et al. (2012)
  - Kopytov et al. (2024): the one we will study here
  - Liu (2019), Liu and Tsyvinski (2024)

Literature (2)

- Firm-to-firm
  - Discrete[2]
    - Dhyne et al. (2023), Dhyne et al. (2022), Carvalho et al. (2020)
  - Continuous[3]
    - Lim (2018), Miyauchi (2024), Huneeus (2020), Eaton et al. (2023)
  - Empirics
    - Dhyne et al. (2020), Bernard et al. (2019)[4], Baqaee et al. (2023)
  - Pure theory (or mainly theory)
    - Oberfield (2018)[5], Acemoglu and Tahbaz-Salehi (2024), Grossman et al. (2023), Grossman et al. (2024a),[6] Grossman et al. (2024b)

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[2] Only quantitative models listed

[3] Only quantitative models listed

[4] Theory and its test

[5] Can be relabeled as sector-to-sector, though.

[6] They calibrated their model, but did not use data on firm-to-firm trade.

Trade-off

- Which would you source from?
  - an expensive, but stable supplier,
  - a cheap, but unstable supplier.

Overview of the model

- Here we study the model of Kopytov, Mishra, Nimark, and Taschereau-Dumouchel (2024).
- There is one representative firm in each sector.
- The representative firm chooses the exponents in its production function.
  - To what extent does the firm rely on each sector?
- The firm is owned by households. As such, it takes risks into account.
  - The intermediate good from this sector enhances my production, but the productivity of this sector is volatile...
- The unique equilibrium is (ex-ante) efficient.
- Therefore, we can characterize equilibrium networks and allocations as a solution to the planner's problem.
- Eventually, the firm's problem reduces to choosing Domar weights.

Setup

- There are n sectors, indexed by $i \in \{1, \cdots, n\}$.
- Each sector produces a differentiated good.
- In each sector, there is a representative firm.
  - We use sector i, firm i, and product i interchangeably.
- Firms face perfect competition. Equilibrium profits are zero.

Production functions and techniques

- Representative firm i has access to a set of production techniques $\mathcal{A}_i$ and chooses only one technique $\alpha_i \in \mathcal{A}_i$.
- The production function of i is

$$F(\alpha_i, L_i, X_i) = e^{\varepsilon_i} A_i(\alpha_i)\zeta(\alpha_i) L_i^{1-\sum_{j=1}^{n}\alpha_{ij}} \prod_{j=1}^{n} X_{ij}^{\alpha_{ij}}, \qquad (1)$$

  - L_i is labor inputs,
  - $X_i = (X_{i1}, \cdots, X_{in})^\top$ is a vector of intermediate inputs,
  - ε_i is the stochastic component of firm i's total factor productivity,
  - $\alpha_i = (\alpha_{i1}, \cdots, \alpha_{in})^\top \in \mathcal{A}_i$ is a production technique that determines intermediate input shares and affects total factor productivity through $A_i(\alpha_i)$,
  - $A_i(\alpha_i)$ is a productivity shifter,
  - $\zeta(\alpha_i)$ is just a normalization to simplify the cost function.[7]

[7] $[\zeta(\alpha_i)]^{-1} = (1 - \sum_{j=1}^{n}\alpha_{ij})^{1-\sum_{j=1}^{n}\alpha_{ij}} \prod_{j=1}^{n} \alpha_{ij}^{\alpha_{ij}}$.

Production techniques and intermediate input shares

- We define i's set of feasible production techniques as

$$\mathcal{A}_i = \{\alpha_i \in [0,1]^n : \sum_{j=1}^{n} \alpha_{ij} \leq \bar{\alpha}_i\},$$

  where $0 < 1 - \bar{\alpha}_i < 1$ is the lower bound of the share of labor (and the upper bound of the sum of the share of intermediate inputs).
- Define $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ (a Cartesian product).
  - $\alpha \in \mathcal{A}$ is a production network in this economy.
  - α represents to what extent each sector relies on intermediate inputs from other sectors.
  - α can be expressed as a matrix.

Productivity shifter $A_i(\alpha_i)$ (1)

- The production technique α_i influences i's total factor productivity through $A_i(\alpha_i)$.
- The authors' example: "beach towels and flowers are not very useful when making a car, and a technique that relies only on these inputs would have a low A_i."

Assumption 1

$A_i(\alpha_i)$ is smooth and strictly log-concave.

- I interpret "smooth" as A_i is differentiable as many times as we wish.
- Let M be a convex subset of R^n. Function $f : M \to R_+$ is strictly log-concave if

$$f(\theta x + (1-\theta)y) > f(x)^{\theta} f(y)^{1-\theta},$$

  for any $x, y \in M$ and $0 < \theta < 1$.
- Note that then $\log f$ is strictly concave if f is strictly positive.

Productivity shifter $A_i(\alpha_i)$ (2)

Why do we need Assumption 1?

1. There exists a unique technique that solves the optimization problem of the firm.
2. For each sector i, there is a unique vector of *ideal* input shares α_i° that maximizes A_i.
   - This represents the most productive way to combine intermediate goods to produce good i.
   - But, this is not necesarily i's technique choice in equilibrium.
   - Why? Because maximizing A_i is not the same as maximizing i's risk-adjusted expected profits.[8]

Without loss of generality, normalize $A_i(\alpha_i^\circ)$ for all i.

[8] This is at the core of this paper. We will see it later.

Productivity shifter $A_i(\alpha_i)$ (3)

Example

One example of a function $A_i(\alpha_i)$ that satisfies Assumption 1 is the quadratic form

$$\log A_i(\alpha_i) = \frac{1}{2}(\alpha_i - \alpha_i^\circ)^\top \bar{H}_i(\alpha_i - \alpha_i^\circ), \quad (2)$$

where $\bar{H}_i$ is a negative-definite matrix that is also the Hessian of $\log A_i$.

Sectoral productivity shocks ε_i

- Let $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)^\top$ be a vector of sectoral productivity shocks.
- We assume that the vector is normally distributed $\varepsilon \sim \mathcal{N}(\mu, \Sigma)$.
- μ determines the expected levels of sectoral productivities.
- Σ determines uncertainty of individual elements of ε and their correlations.
- ε is the only source of uncertainty in this economy.
- Each firm i chooses α_i before ε is realized.
- A high μ_i leads to a low unit cost and a low price of good i.
- A high Σ_{ii} leads to a volatile price of good i.
- A high Σ_{ij} leads to more correlated prices of good i and j.
- These affect the sourcing decisions of the firms.

Households (1)

- There is one risk-averse representative household in this economy.
- She chooses $C = (C_1, \cdots, C_n)$ to maximize

$$u\left(\left(\frac{C_1}{\beta_1}\right)^{\beta_1} \times \cdots \times \left(\frac{C_n}{\beta_n}\right)^{\beta_n}\right), \tag{3}$$

  where $\beta_i > 0$ for all i and $\sum_{i=1}^{n} \beta_i = 1$.
- We refer to $Y = \prod_{i=1}^{n}(\beta_i^{-1} C_i)^{\beta_i}$ as aggregate consumption or GDP.
- The utility function $u(\cdot)$ is CRRA with coefficient of relative risk aversion ρ. That is, (3) is rewritten as

$$\frac{Y^{1-\rho}}{1-\rho}.$$

Households (2)

- The household makes consumption decisions after uncertainty is resolved.
- In each state of the world, the household faces the budget constraint

$$\sum_{i=1}^{n} P_i C_i \leq 1,$$

where P_i is the price of good i, and the wage is used as a numeraire.

- Firms are owned by the representative household.
- Firms maximize expected profits discounted by the household's stochastic discount factor

$$\Lambda = u'(Y)/\bar{P}, \tag{4}$$

where $\bar{P} = \prod_{i=1}^{n} P_i^{\beta_i}$.

Households (3)

- By solving the household's optimization, we can show that

$$y = -\beta^\top p$$

  where $y = \log Y$, $\beta = (\beta_1, \cdots, \beta_n)^\top$ and $p = (p_1, \cdots, p_n)^\top$.
- For any i, $p_i = \log P_i$.

Representative firms' optimization: two steps

1. The firm decides which production technique to use.
   - This choice is made before the random productivity vector ε is realized.
2. The firm chooses labor and intermediate inputs after the realization of ε.
   - And the household chooses consumption after the realization of ε.
   - That is, the final demand for each good is also determined after the realization of ε.

We solve these problems backwardly.

The firm's second-stage problem

- Under a given technique α_i, the cost minimization problem of firm i is

$$K_i(\alpha_i, P) = \min_{L_i, X_i} \left(L_i + \sum_{j=1}^{n} P_j X_{ij} \right), \tag{5}$$

subject to

$$F(\alpha_i, L_i, X_i) \geq 1.$$

- The solution to this problem implicitly defines the unit cost of production $K_i(\alpha_i, P)$.
- Using the production function (1), the unit cost function is

$$K_i(\alpha_i, P) = \frac{1}{e^{\varepsilon_i} A_i(\alpha_i)} \prod_{j=1}^{n} P_j^{\alpha_{ij}}. \tag{6}$$

The firm's first-stage problem

- Given an expression for K_i, the first stage of the representative firm's problem is to pick a technique $\alpha_i \in \mathcal{A}_i$ to maximize expected discounted profits

$$\alpha_i^* \in \arg\max_{\alpha_i \in \mathcal{A}_i} E\left[\Lambda Q_i (P_i - K_i(\alpha_i, P))\right]. \tag{7}$$

  - Q_i is the equilibrium demand for good i,
  - the profits in different states of the world are weighted by the household's stochastic discount factor Λ.
- The representative firm takes P, Q_i, and Λ as given.
- Therefore, this problem reduces to $\min_{\alpha_i \in \mathcal{A}_i} E[\Lambda Q_i K_i(\alpha_i, P)]$.
- The firm minimizes the weighted expectation of the total cost $Q_i K_i(\alpha_i, P)$ with the weights being Λ.
- The firm inherits the risk attitude of the representative household.

Equilibrium prices

- In equilibrium, competitive pressure pushes prices to be equal to unit costs

$$P_i = K_i(\alpha_i, P) \tag{8}$$

for all $i \in \{1, \cdots, n\}$.

Definition 1 (equilibrium)

An equilibrium is a choice of technique $\alpha^* = (\alpha_1^*, \cdots, \alpha_n^*)$ and a stochastic tuple $(P^*, C^*, L^*, X^*, Q^*)$ such that:

1. (optimal technique choice) For each $i \in \{1, \cdots, n\}$, the technique choice $\alpha_i^* \in \mathcal{A}_i$ solves (7) given prices P^*, demand Q_i^*, and the stochastic discount factor Λ^* given by (4).
2. (Optimal input choice) For each $i \in \{1, \cdots, n\}$, factor demands per unit of output L_i^*/Q_i^* and X_i^*/Q_i^* are a solution to (5) given prices P^* and the chosen technique α_i^*.
3. (Consumer maximization) The consumption vector C^* maximizes (3).

Definition 1 (equilibrium): continued

4. (Unit cost pricing) For each $i \in \{1, \cdots, n\}$, P_i^* solves (8) where $K_i(\alpha_i^*, P^*)$ is given by (6).
5. (Market clearing) For each $i \in \{1, \cdots, n\}$,

$$C_i^* + \sum_{j=1}^{n} X_{ji}^* = Q_i^* = F_i(\alpha_i^*, L_i^*, X_i^*), \text{ and } \sum_{i=1}^{n} L_i^* = 1.$$

Comments on Definition 1

- Conditions 2-5 correspond to the standard competitive equilibrium conditions for an economy with a fixed production network.
  - Firms and the household optimize in a competitive environment.
  - All markets clear given equilibrium prices.
- Condition 1 emphasizes that production techniques, and hence the production network represented by the matrix α^*, are equilibrium objects.

How we proceed

- Fixed production networks,
- Endogenous production networks.

Two objects: the Leontief inverse and the Domar weight

- The Leontief inverse is

$$\mathcal{L}(\alpha) = (I - \alpha)^{-1} = I + \alpha + \alpha^2 + \cdots ,$$

  where

$$\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix},$$

  and I is the $n \times n$ identity matrix.
- Define the Domar weight ω_i of sector i as the ratio of its sales to nominal GDP

$$\omega_i = \frac{P_i Q_i}{P^\top C}.$$

  - The vector of Domar weights $\omega = (\omega_1, \cdots, \omega_n)^\top$ satisfies $\omega^\top = \beta^\top \mathcal{L}(\alpha) > 0$.

Lemma 1

Under a given network α, the vector of log prices is given by

$$p(\alpha) = -\mathcal{L}(\alpha)(\varepsilon + a(\alpha)),$$

and log GDP is given by

$$y(\alpha) = \omega(\alpha)^\top(\varepsilon + a(\alpha)),$$

where $a(\alpha) = (\log A_1(\alpha_1), \cdots, \log A_n(\alpha_n))^\top$.

- Lemma 1 describes how prices and GDP depend on the productivity vector $\varepsilon + a(\alpha)$ and the production network α.
- An increase in productivity pushes down prices through the Leontief matrix $\mathcal{L}(\alpha)$.
- An increase in productivity has a linear and positive effect on GDP with the coefficient being the Domar weight.

Mean and variance of GDP

Under a fixed network α,

$$E[y(\alpha)] = \omega(\alpha)^\top(\mu + a(\alpha)) \tag{9}$$

and

$$V[y(\alpha)] = \omega(\alpha)^\top \Sigma \omega(\alpha). \tag{10}$$

Corollary 1

For a fixed production network α, the following hold

1. The impact of a change in expected TFP mu_i on the moments of log GDP is given by

$$\frac{\partial E[y]}{\partial \mu_i} = \omega_i, \text{ and } \frac{\partial V[y]}{\partial \mu_i} = 0.$$

2. The impact of a change in volatility Σ_{ij} on the moments of log GDP is given by

$$\frac{\partial E[y]}{\partial \Sigma_{ij}} = 0, \text{and } \frac{\partial V[y]}{\partial \Sigma_{ij}} = \omega_i \omega_j.$$

- The first part is the Hulten theorem.
- For the second part, first think about the case $j = i$.

Firm decisions (1)

- We have discussed the case of fixed production networks.
- Now we move on to endogenous production networks.
- Let α^* be the equilibrium network.
- Let $\lambda(\alpha^*) = \log \Lambda(\alpha^*)$ (the log of the stochastic discount factor).
- Let $k_i(\alpha_i, \alpha^*) = \log K_i(\alpha_i, P^*(\alpha^*))$ (the log of the unit cost).
- Using these notations, we can reorganize firm i's maximization problem (7) as

$$\alpha_i^* \in \arg\min_{\alpha_i \in \mathcal{A}_i} E[k_i(\alpha_i, \alpha^*)] + Cov[\lambda(\alpha^*), k_i(\alpha_i, \alpha^*)]. \quad (11)$$

  - We can rewrite like this because $\lambda(\alpha^*)$, $p_i(\alpha^*)$, and $k_i(\alpha_i, \alpha^*)$ are normally distributed.
  - See the supplementary material for details.

Firm decisions (2)

- Taking the expected value of the log of (6), we have

$$E[k_i(\alpha_i, \alpha^*)] = -\mu_i - a_i(\alpha_i) + \sum_{j=1}^{n} \alpha_{ij} E[p_j].$$

- That is, the firm prefers techniques that have high productivity a_i and that rely on inputs that are expected to be cheap.
- The second term in (11) captures the importance of aggregate risk for the firm's decision.
  - The firm prefers to have a low unit cost in states of the world in which the marginal utility of consumption is high.
  - The economy is in a bad situation.
  - $\Rightarrow$ Aggregate consumption (GDP) is low.
  - $\Rightarrow$ The marginal utility is low.
  - $\Rightarrow$ The firm really wants to have low costs in such a situation.

Lemma 2

In equilibrium, the technique choice problem of the representative firm in sector i is

$$\alpha_i^* \in \arg\max_{\alpha_i \in \mathcal{A}_i} a_i(\alpha_i) - \sum_{j=1}^{n} \alpha_{ij} \mathcal{R}_j(\alpha^*), \qquad (12)$$

where

$$\mathcal{R}(\alpha^*) = E[p(\alpha^*)] + Cov[p(\alpha^*), \lambda(\alpha^*)]$$

is the vector of equilibrium risk-adjusted prices, and where

$$E[p(\alpha^*)] = -\mathcal{L}(\alpha^*)(\mu + a(\alpha^*))$$

and

$$Cov[p(\alpha^*), \lambda(\alpha^*)] = (\rho - 1)\mathcal{L}(\alpha^*)\Sigma[\mathcal{L}(\alpha^*)]^\top \beta.$$

Comments on Lemma 2

- All the equilibrium information needed for the firm's problem is contained in the vector of risk-adjusted prices $\mathcal{R}$.
- $\mathcal{R}$ quantity provides an overall measure of the desirability of an input that depends on its expected price and on how its price covaries with the stochastic discount factor.
- Goods that are cheap when aggregate consumption is low are particularly attractive as inputs, controlling for expected prices.

The Hessian matrix of a_i

- Define H_i by the Hessian matrix of a_i

$$H_i = \begin{bmatrix} \frac{\partial^2 a_i}{\partial \alpha_{i1}^2} & \frac{\partial^2 a_i}{\partial \alpha_{i1} \partial \alpha_{i2}} & \cdots & \frac{\partial^2 a_i}{\partial \alpha_{i1} \partial \alpha_{in}} \\ \frac{\partial^2 a_i}{\partial \alpha_{i2} \partial \alpha_{i1}} & \frac{\partial^2 a_i}{\partial \alpha_{i2}^2} & \cdots & \frac{\partial^2 a_i}{\partial \alpha_{i2} \partial \alpha_{in}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 a_i}{\partial \alpha_{in} \partial \alpha_{i1}} & \frac{\partial^2 a_i}{\partial \alpha_{in} \partial \alpha_{i2}} & \cdots & \frac{\partial^2 a_i}{\partial \alpha_{in}^2} \end{bmatrix}.$$

- Taking the first order condition of (12) and applying the implicit function theorem, we can show that

$$\frac{\partial \alpha_{ij}}{\partial \mathcal{R}_k} = [H_i^{-1}(\alpha_i)]_{jk},$$

where $[\cdot]_{jk}$ denotes the (j, k) element of a matrix.

Complements vs substitutes: the Hessian matters

- Does an increase in k's risk-adjusted price $\mathcal{R}_k$ lead to a decrease or an increase in the share of another input $j \neq k$?
- If $[H_i^{-1}]_{jk} > 0$, we say j and k are substitutes (for i).
  - $\mathcal{R}_k \uparrow \Rightarrow \alpha_{ij} \uparrow$
  - Substituting away from k to j
- If $[H_i^{-1}]_{jk} < 0$, we say j and k are complements (for i).
  - $\mathcal{R}_k \uparrow \Rightarrow \alpha_{ij} \downarrow$
  - If k is too expensive to buy, i doesn't need j either
- One sufficient condition for a Hessian matrix H_i to exhibit complementarity for all sectors is $[H_i]_{jk} \geq 0$ for all $j \neq k$.

Example: 4 sectors in partial equilibrium

Input 1 Steel

Input 2 Equipment

- Milling machines and lathes to transform raw steel into usable components

Input 3 Carbon fiber

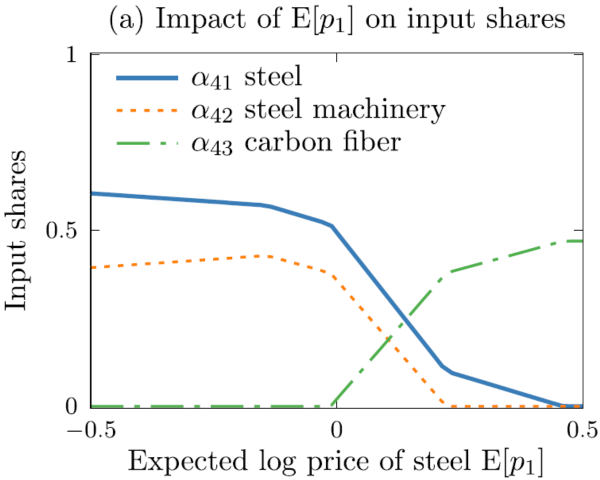
- Carbon fiber can replace steel

Sector 4 Car manufacturing

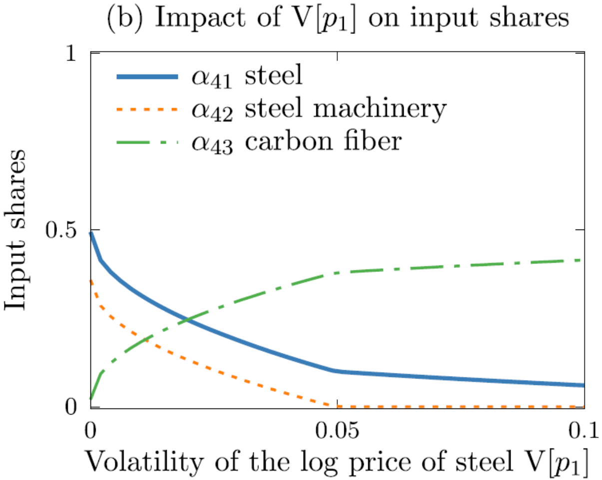
- This car manufacturer has a TFP shifter function

$$
\begin{aligned}
a_4(\alpha_4) = &-\sum_{j=1}^{4} \kappa_j(\alpha_{4j} - \alpha_{4j}^{\circ})^2 - \psi_1(\alpha_{41} - \alpha_{42})^2 \\
&- \psi_2[(\alpha_{41} + \alpha_{43}) - (\alpha_{41}^{\circ} + \alpha_{43}^{\circ})^2].
\end{aligned}
$$

Input shares and $E[p_1]$



Input shares and $V[p_1]$



Lemma 3

An efficient production network α^* solves

$$\mathcal{W} := \max_{\alpha \in \mathcal{A}} W(\alpha, \mu, \sigma),$$

where is a measure of the welfare of the household, and where

$$W(\alpha, \mu, \sigma) := E[y(\alpha)] - \frac{1}{2}(\rho - 1)V[y(\alpha)] \tag{13}$$

is welfare under a given network α.

- The relative risk aversion ρ determines the relative importance of the expected log GDP $E[y(\alpha)]$ and the variance of the log GDP $V[y(\alpha)]$.
- The welfare depends on only first and second moments of log GDP. This is because preferences are CRRA and log GDP (aggregate consumption) is normally distributed.

Toward Domar weights (1)

- Using (9) and (10), we can rewrite the objective function (13) as
$$\omega^\top \mu + \omega^\top a(\alpha) - \frac{1}{2}(\rho - 1)\omega^\top \Sigma \omega.$$
- Remember that μ and Σ are exogenous parameters.
  - So the social planner cannot choose them.
- Only endogenous variables are ω and $a(\alpha)$.
- Moreover, the only term that does not depend exclusively on ω is $\omega^\top a(\alpha)$.
- We want to rewrite this in terms of ω alone.

Toward Domar weights (2)

- Consider the optimization problem

$$\bar{a}(\omega) := \max_{\alpha \in \mathcal{A}} \omega^\top a(\omega), \tag{14}$$

subject to the definition of the Domar weights given by $\omega^\top = \beta^\top \mathcal{L}(\alpha)$.

- We refer to the value function $\bar{a}$ as the aggregate TFP shifter function.
- We denote by $\alpha(\omega)$ the solution to (14).
- For given ω, $\bar{a}$ and $\alpha(\omega)$ do not depend on μ or Σ.

Example

- We can explicitly solve for $\bar{a}$ and $\alpha(\omega)$ under the quadratic TFP shifter function (2).
- At an interior solution $\alpha \in \mathrm{int}\mathcal{A}$, the optimal production network $\alpha(\omega)$ for a given vector of Domar weights ω satisfies

$$\alpha_i(\omega) - \alpha_i^\circ = H_i^{-1} \left(\sum_{j=1}^{n} \omega_j H_j^{-1} \right)^{-1} \left(\omega - \beta - \sum_{j=1}^{n} \omega_j \alpha_j^\circ \right)$$

  for all i, and the associated value function $\bar{a}$ is

$$\bar{a}(\omega) = \frac{1}{2} \sum_{i=1}^{n} \omega_i (\alpha_i(\omega) - \alpha_i^\circ)^\top H_i (\alpha_i(\omega) - \alpha_i^\circ).$$

- The gradients ∇a_i of the TFP shifter functions are all equal to each other such that

$$\nabla a_i = \nabla a_j$$

  for all i, j.

Corollary 2

The efficient Domar weight vector ω^* solves

$$\mathcal{W} = \max_{\omega \in \mathcal{O}} \underbrace{\omega^\top \mu + \bar{a}(\omega)}_{E[y]} - \frac{1}{2}(\rho - 1) \underbrace{\omega^\top \Sigma \omega}_{V[y]}, \tag{15}$$

where $\mathcal{O} = \{\omega \in R^n_+ : \omega \geq \beta \text{ and } 1 \geq \omega^\top (\mathbf{1} - \bar{\alpha})\}$ and $\bar{\alpha}(\omega)$ is given by (14).

- The set $\mathcal{O}$ contains the vectors ω such that the corresponding production network $\alpha(\omega) \in \mathcal{A}$.
- The first inequality follows from $\alpha_{ij} \geq 0$ for all i, j.
- The second inequality, where $\mathbf{1}$ denotes the $n \times 1$ all-one column vector, follows from $\sum_j \alpha_{ij} \leq \bar{\alpha}_i$ for all i.

Lemma 4

The objective function of the planner's problem (15) is strictly concave. Furthermore, there is a unique vector of Domar weights ω^* that solves that problem, and there is a unique production network $\alpha(\omega^*)$ associated with that solution.

- Therefore, the first-order conditions will characterize the unique unique efficient network.

Proposition 1

There exists a unique equilibrium, and it is efficient.

Taking stock

- There is a unique equilibrium, and it is efficient.
- Finding the efficient network reduces to finding the Domar weights associated with the network.
- Therefore, finding the equilibrium network reduces to finding the efficient Domar weights.

"Beliefs"

- Somehow, in the paper, the authors call μ and Σ "beliefs."
- They are just the mean vector and the covariance matrix of the stochastic part of (log) TFP.
- Maybe they refer to how households and producers "perceive" the level and uncertainty of productivity.
- We will look at some of the results in the paper about how beliefs affect equilibrium outcomes.

Impacts of beliefs

1. Impacts on Domar weights,
2. Impacts on welfare.

Proposition 2

The Domar weight ω_i of sector i is (weakly) increasing in μ_i and (weakly) decreasing in Σ_{ii}.

Risk-adjusted productivity shocks

- We define a risk-adjusted version of the productivity vector ε

$$\mathcal{E} = \underbrace{\mu}_{E[\varepsilon]} - \underbrace{(\rho - 1)\Sigma\omega}_{Cov[\varepsilon,\lambda]}.$$

- This measures how higher exposure to ε affects the household's utility.
- Remember that λ denotes the log of the stochastic discount factor Λ.
- Let $\mathbf{1}_i$ be the column vector with a 1 only in the i-th element and zeros otherwise. Then

$$\frac{\partial \mathcal{E}}{\partial \mu_i} = \mu_i,$$

and

$$\frac{\partial \mathcal{E}}{\partial \Sigma_{ij}} = -\frac{1}{2}(\rho - 1)(\omega_j \mathbf{1}_i + \omega_i \mathbf{1}_j).$$

Proposition 3

Let γ denote either μ_i or Σ_{ij}. If $\omega \in \text{int}\mathcal{O}$, then

$$\frac{d\omega}{d\gamma} = \underbrace{-\mathcal{H}^{-1}}_{propagation} \times \underbrace{\frac{\partial \mathcal{E}}{\partial \gamma}}_{impulse},$$

where the $n \times n$ negative definite matrix $\mathcal{H}$ is given by

$$\mathcal{H} = \nabla^2 \bar{a} + \frac{d\mathcal{E}}{d\omega},$$

and where the matrix $\nabla^2 \bar{a}$ is the Hessian of the aggregate TFP shifter function $\bar{a}$, and $\frac{d\mathcal{E}}{d\omega} = -\frac{dCov[\varepsilon, \lambda]}{d\omega} = -(\rho - 1)\Sigma$ is the Jacobian matrix of the risk adjusted TFP vector $\mathcal{E}$.

Comments on Proposition 3

- The impulse (the 2nd part on the RHS) captures the direct effect on the risk-adjusted TFP.
- The propagation (the 1st part on the RHS) captures the global, economy-wide substitution patterns between sectors.
  - Contrast it with H_i^{-1} (local, firm-level substitutution).
- If $\mathcal{H}_{ij}^{-1} < 0$, i and j are global complements.
  - $\mathcal{E}_i \uparrow \Rightarrow \omega_j \uparrow$
- If $\mathcal{H}_{ij}^{-1} > 0$, i and j are global substitutes.
  - $\mathcal{E}_i \uparrow \Rightarrow \omega_j \downarrow$

What's $\mathcal{H}$?

$$\mathcal{H} = \nabla^2 \bar{a} - \underbrace{(\rho - 1)\Sigma}_{\frac{dCov[\varepsilon,\lambda]}{d\omega}}$$

Two forces:

1. Aggregate TFP shifter function $\bar{a}$
   - Local substitution patterns in $(a_1, \cdots, a_n)$ contribute to global substitution patterns
2. Covariance matrix Σ
   - Suppose that ω_i increases because of a positive shock in i.
   - In response to an increase in ω_i, the planner puts a lower ω_j as Σ_{ij} increases.

$$\frac{\partial \mathcal{H}^{-1}_{ij}}{\partial \Sigma_{ij}} > 0$$

Proposition 5

Let denote γ either μ_i or Σ_{ij}. Under an endogenous network, welfare responds to a marginal change in γ as if the network were fixed at its equilibrium value α^*, that is

$$\frac{d\mathcal{W}(\mu, \Sigma)}{d\gamma} = \frac{\partial W(\alpha^*, \mu, \Sigma)}{\partial \gamma}.$$

How about non-infinitesimal change?

- Let $\alpha^*(\mu, \Sigma)$ be the equilibrium production network under (μ, Σ).

$$\underbrace{\mathcal{W}(\mu', \Sigma') - \mathcal{W}(\mu, \Sigma)}_{\text{Change in welfare under a flexible network}}$$
$$\geq \underbrace{W(\alpha^*(\mu, \Sigma), \mu', \Sigma') - W(\alpha^*(\mu, \Sigma), \mu, \Sigma)}_{\text{Change in welfare under a fixed network}}.$$

Corollary 4

The impact of an increase in μ_i on welfare is given by

$$\frac{d\mathcal{W}}{d\mu_i} = \omega_i,$$

and the impact of an increase in Σ_{ij} on welfare is given by

$$\frac{d\mathcal{W}}{d\Sigma_{ij}} = -\frac{1}{2}(\rho - 1)\omega_i\omega_j.$$

- This is a direct result from Corollary 1, Proposition 5, and (13).

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