Term Paper: Answer Key

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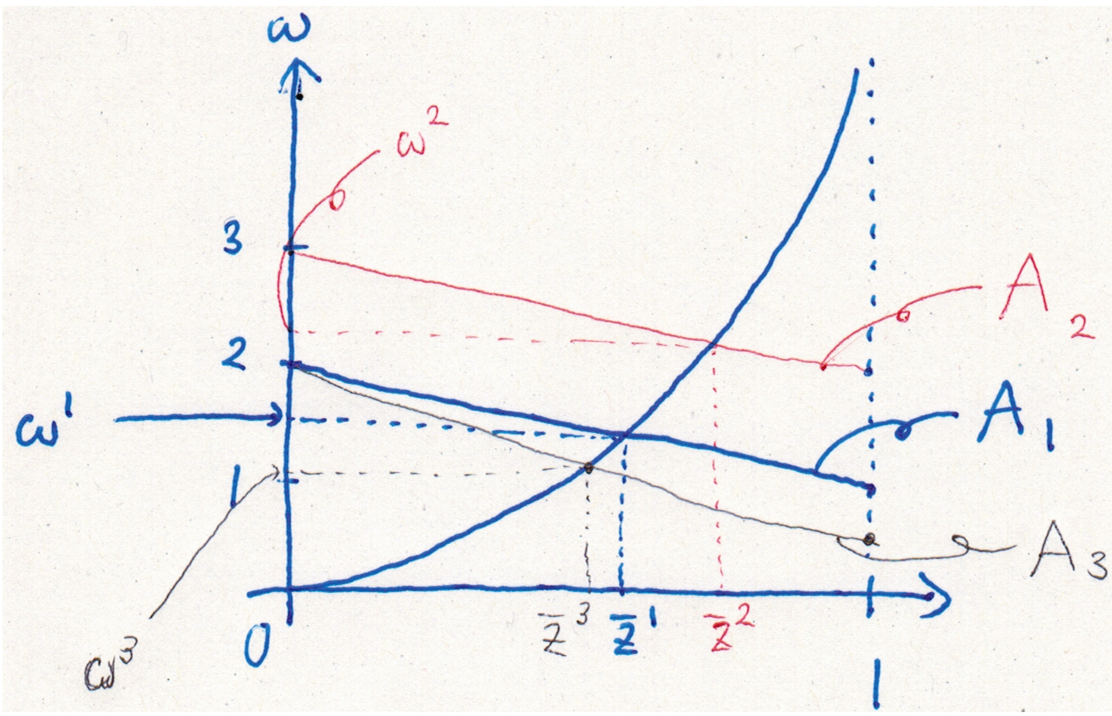
1 The Dornbusch-Fischer-Samuelson model

Figure 1: Comparative statics with respect to A



Recall that the equilibrium relative wage $\bar{\omega}$ and the equilibrium cutoff (of goods) $\bar{z}$ are characterized as the intersection of the downward-sloping A curve and the upward-sloping B curve. B doesn't change throughout this exercise.

Initially, the relative unit labor requirement A is $A_1 = 2 - z$. Now suppose that this shifts up to $A_2 = 3 - z$. Remember that $A(z) = a^*(z)/a(z)$, where $a^*(z)$ and $a^*(z)$ are Foreign and Home's unit labor requirement to produce good z, respectively. Therefore, for all goods, Foreign becomes less productive. Then the cutoff shifts from $\bar{z}^1$ to $\bar{z}^2$. Recall that the cutoff $\bar{z}$ is such that Home produces goods $[0, \bar{z}]$ and Foreign produces goods $[\bar{z}, 1]$. Therefore, because of this shift of relative unit labor requirement, Home produces a wider range of goods and Foreign produces a narrower range of goods. Accordingly, the relative wage between Home and Foreign increases from ω^1 to ω^2. In short, Home becomes productive uniformly for all goods; Home becomes an exporter of a wider range of goods; and the wage ratio between Home and Foreign increases.

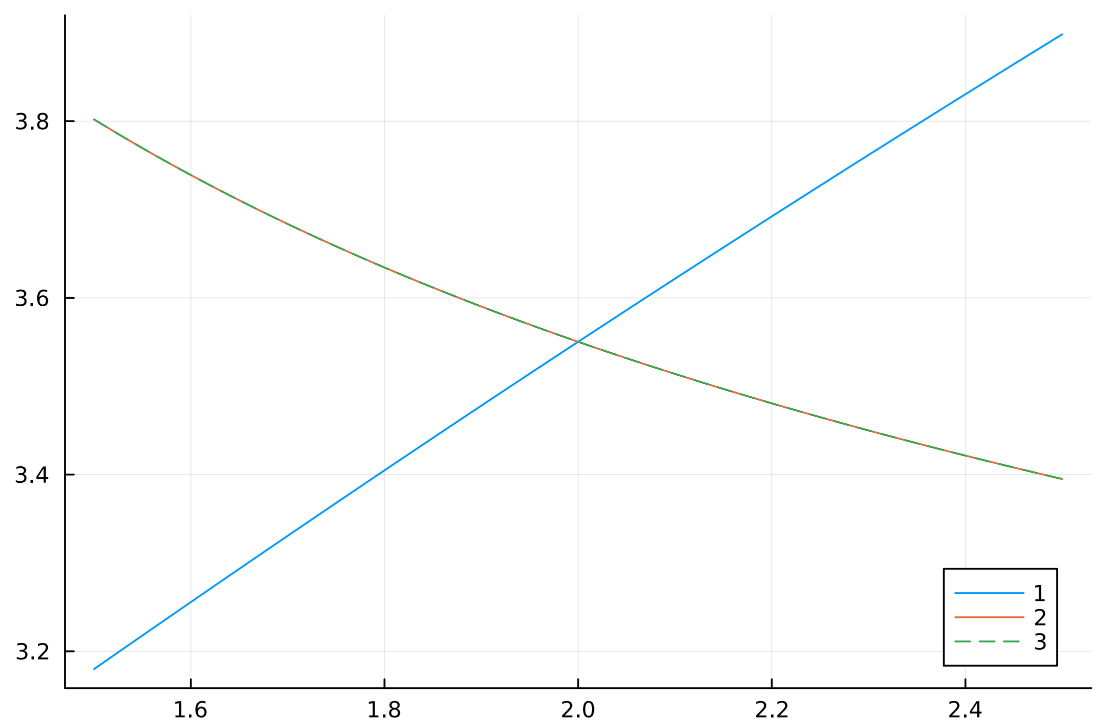
Now the relative unit labor requirement rotates clockwise from A_1 to A_2. For any $z \in (0, 1]$, the relative unit labor requirement decreases. This means that for all goods except good 0, Home becomes relatively less productive. Therefore, the cutoff decreases from $\bar{z}^1$ to $\bar{z}^3$. In short, Home becomes less productive for almost all goods; Home becomes an importer of a wider range of goods; and the relative wage between Home and Foreign decreases.

2 Computing the Eaton-Kortum model

2.1 Comparative statics with respect to T_1

1. See Figure 2.

Figure 2: Q2.1.1



2. See Figure 3.

2.2 Comparative statics with respect to $d_{12} = d_{21}$

1. See Figure 4.

2. See Figure 5.

3. See Figure 6.

Figure 3: Q2.1.2

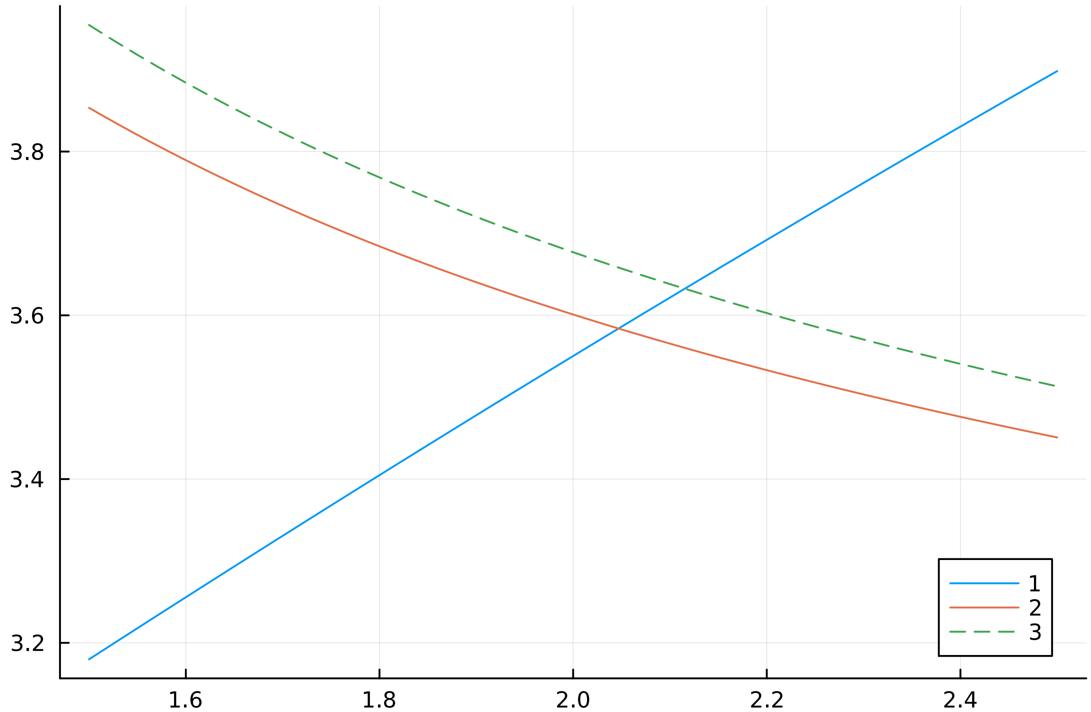


Figure 4: Q2.2.1

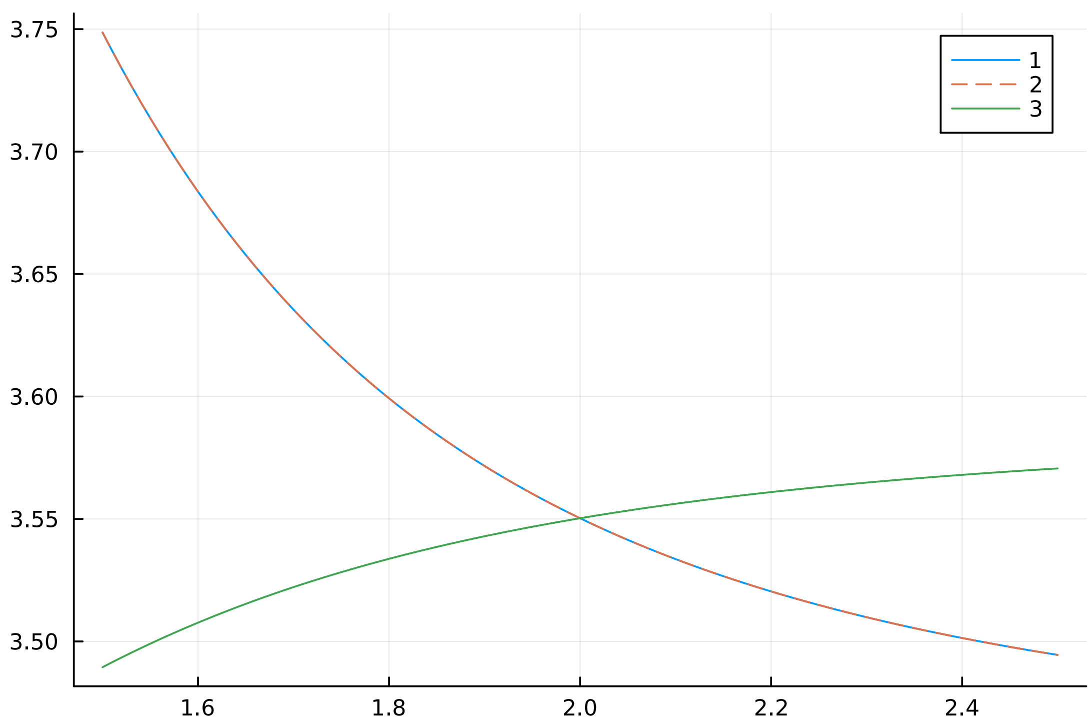


Figure 5: Q2.2.2

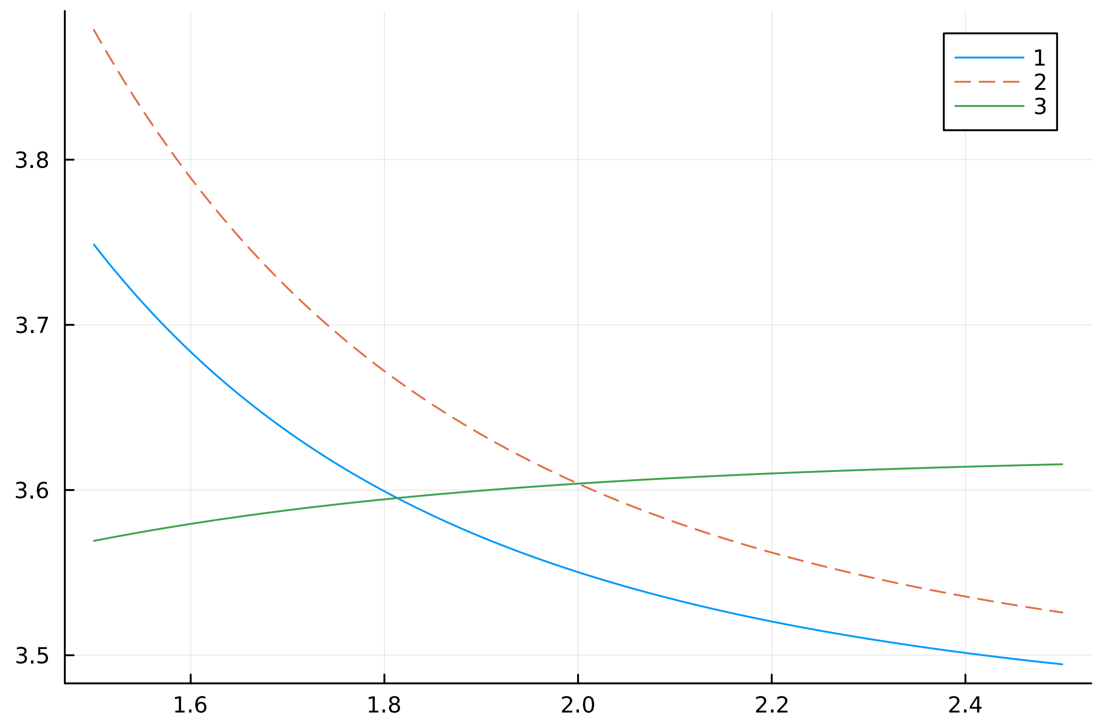
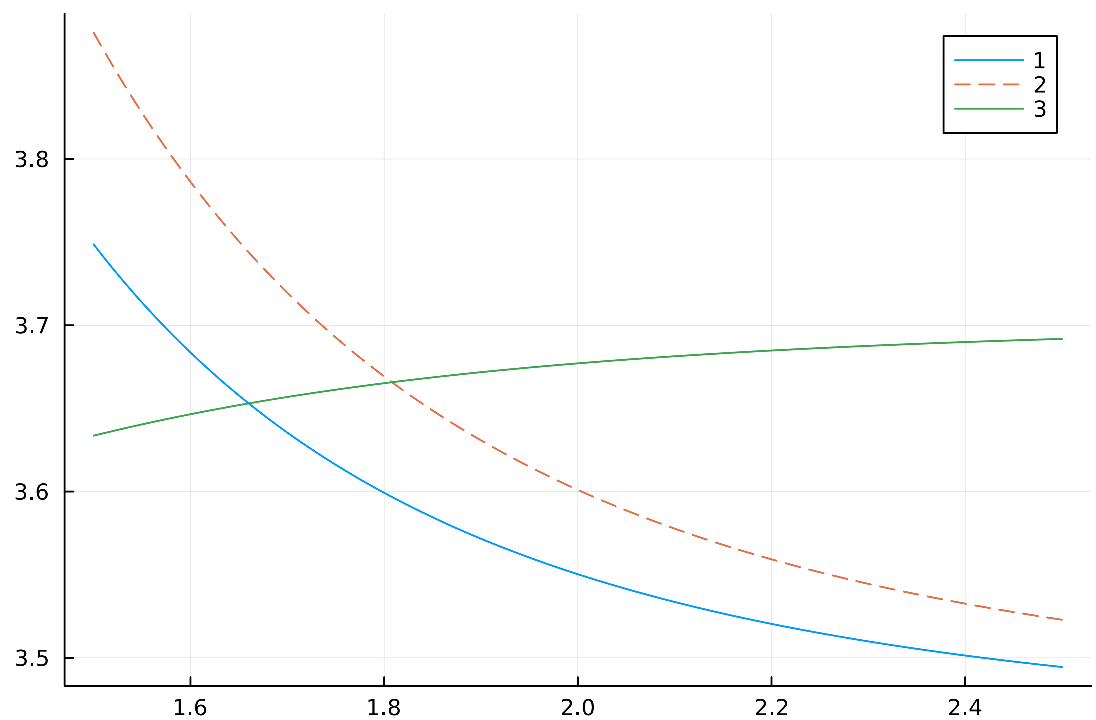


Figure 6: Q2.2.3



3 The Melitz-Chaney model

1. The firm's profit-maximization problem is

$$\max_{\{p_{ij}(\varphi)\}_{j\in S}} \sum_{j\in S} \left(p_{ij}(\varphi)^{1-\sigma} Y_j P_j^{\sigma-1} - \frac{w_i}{\varphi}\tau_{ij} p_{ij}(\varphi)^{-\sigma} Y_j P_j^{\sigma-1} - f_{ij} \right).$$

The first-order condition with respect to $p_{ij}(\varphi)$ is

$$(1-\sigma)p_{ij}(\varphi)^{-\sigma} Y_j P_j^{\sigma-1} + \sigma \frac{w_i}{\varphi}\tau_{ij} p_{ij}(\varphi)^{-\sigma-1} Y_j P_j^{\sigma-1} = 0.$$

Rearranging this, we have

$$\sigma \frac{w_i}{\varphi}\tau_{ij} p_{ij}(\varphi)^{-\sigma-1} = (\sigma-1) p_{ij}(\varphi)^{-\sigma}.$$

Solving this for $p_{ij}(\varphi)$,

$$p_{ij}(\varphi) = \frac{\sigma}{\sigma-1}\frac{w_i}{\varphi}\tau_{ij}.$$

2. Our starting point is

$$f_i^e = \sum_{j\in S} \int_{\varphi_{ij}^*}^{\infty} (\pi_{ij}(\varphi) - f_{ij}) dG_i(\varphi).$$

Substituting $\pi_{ij}(\varphi) = \frac{1}{\sigma}\left(\frac{\sigma}{\sigma-1}\frac{w_i}{\varphi}\tau_{ij}\right)^{1-\sigma} Y_j P_j^{\sigma-1}$ and $g_i(\varphi) = \theta_i \varphi^{-(\theta_i+1)}$ into this, we have

$$\begin{aligned} f_i^e &= \sum_{j\in S} \int_{\varphi_{ij}^*}^{\infty} \left[\frac{1}{\sigma}\left(\frac{\sigma}{\sigma-1}\frac{w_i}{\varphi}\tau_{ij}\right)^{1-\sigma} Y_j P_j^{\sigma-1} - f_{ij} \right] \theta_i \varphi^{-(\theta_i+1) d\varphi} \\ &= \sum_{j\in S} \int_{\varphi_{ij}^*}^{\infty} \left[\frac{\theta_i}{\sigma}\left(\frac{\sigma}{\sigma-1} w_i\tau_{ij}\right)^{1-\sigma} Y_j P_j^{\sigma-1} \varphi^{\sigma-2-\theta_i} - \theta_i f_{ij} \varphi^{-(\theta_i+1)} \right] d\varphi \\ &= \sum_{j\in S} \left[\frac{1}{\sigma-\theta_i-1}\frac{\theta_i}{\sigma}\left(\frac{\sigma}{\sigma-1} w_i\tau_{ij}\right)^{1-\sigma} Y_j P_j^{\sigma-1} \varphi^{\sigma-\theta_i-1} + \frac{1}{\theta_i}\theta_i f_{ij}\varphi^{-\theta_i} \right]_{\varphi_{ij}^*}^{\infty}. \end{aligned}$$

Since $\theta_i > \sigma - 1$, we have $\sigma - \theta_i - 1 < 0$. Of course we hace $-\theta_i < 0$. Therefore, we have

$$f_i^e = \sum_{j\in S} \left[-\frac{1}{\sigma-\theta_i-1}\frac{\theta_i}{\sigma}\left(\frac{\sigma}{\sigma-1} w_i\tau_{ij}\right)^{1-\sigma} Y_j P_j^{\sigma-1} (\varphi_{ij}^*)^{\sigma-\theta_i-1} - f_{ij}(\varphi_{ij}^*)^{-\theta_i} \right].$$

Substituting $\varphi_{ij}^* = \left(\frac{\sigma f_{ij}\left(\frac{\sigma}{\sigma-1} w_i\tau_{ij}\right)^{\sigma-1}}{Y_j P_j^{\sigma-1}}\right)^{\frac{1}{\sigma-1}}$ into this,

$$f_i^e = \sum_{j\in S} \left[\frac{1}{\theta_i+1-\sigma}\frac{\theta_i}{\sigma}\left(\frac{\sigma}{\sigma-1} w_i\tau_{ij}\right)^{1-\sigma} Y_j P_j^{\sigma-1} \left(\frac{\sigma f_{ij}\left(\frac{\sigma}{\sigma-1} w_i\tau_{ij}\right)^{\sigma-1}}{Y_j P_j^{\sigma-1}}\right)^{\frac{\sigma-\theta_i-1}{\sigma-1}} - f_{ij}\left(\frac{\sigma f_{ij}\left(\frac{\sigma}{\sigma-1} w_i\tau_{ij}\right)^{\sigma-1}}{Y_j P_j^{\sigma-1}}\right)^{-\frac{\theta_i}{\sigma-1}} \right]. \quad (1)$$

The first term in the square bracket is

$$\frac{1}{\theta_i+1-\sigma}\frac{\theta_i}{\sigma}\left(\frac{\sigma}{\sigma-1}w_i\tau_{ij}\right)^{1-\sigma}Y_jP_j^{\sigma-1}(\sigma f_{ij})^{1-\frac{\theta_i}{\sigma-1}}\left(\frac{\sigma}{\sigma-1}w_i\tau_{ij}\right)^{\sigma-\theta_i-1}(Y_jP_j^{\sigma-1})^{-1+\frac{\theta_i}{\sigma-1}}$$
$$=\frac{\theta_i}{\theta_i+1-\sigma}\sigma^{-\frac{\theta_i}{\sigma-1}}f_{ij}^{1-\frac{\theta_i}{\sigma-1}}\left(\frac{\sigma}{\sigma-1}w_i\tau_{ij}\right)^{-\theta_i}(Y_{jP_j^{\sigma-1}})^{\frac{\theta_i}{\sigma-1}}.$$

The second term in the square bracket is

$$(f_{ij})^{1-\frac{\theta_i}{\sigma-1}}\sigma^{-\frac{\theta_i}{\sigma-1}}\left(\frac{\sigma}{\sigma-1}w_i\tau_{ij}\right)^{-\theta_i}(Y_jP_j^{\sigma-1})^{\frac{\theta_i}{\sigma-1}}.$$

Then the right-hand side of (1) is

$$\sum_{j\in S}\left[\frac{\theta_i}{\theta_i+1-\sigma}\sigma^{-\frac{\theta_i}{\sigma-1}}f_{ij}^{1-\frac{\theta_i}{\sigma-1}}\left(\frac{\sigma}{\sigma-1}w_i\tau_{ij}\right)^{-\theta_i}(Y_jP_j^{\sigma-1})^{\frac{\theta_i}{\sigma-1}}-(f_{ij})^{1-\frac{\theta_i}{\sigma-1}}\sigma^{-\frac{\theta_i}{\sigma-1}}\left(\frac{\sigma}{\sigma-1}w_i\tau_{ij}\right)^{-\theta_i}(Y_jP_j^{\sigma-1})^{\frac{\theta_i}{\sigma-1}}\right]$$
$$=\sum_{j\in S}\frac{\sigma-1}{\theta_i+1-\sigma}\left(\frac{\sigma}{\sigma-1}w_i\tau_{ij}\right)^{-\theta_i}\sigma^{-\frac{\theta_i}{\sigma-1}}f_{ij}^{\frac{\sigma-\theta_i-1}{\sigma-1}}Y_j^{\frac{\theta_i}{\sigma-1}}P_j^{\theta_i}.$$

4 Production networks

1. The cost minimization problem is

$$\min_{L_i,X_j}\quad L_i+\sum_{j=1}^{n}P_jX_{ij}, \tag{2}$$
$$\text{s.t.}\quad e^{\varepsilon_i}A_i(\alpha_i)\zeta(\alpha_i)L_i^{\sum_{j=1}^n 1-\alpha_{ij}}\prod_{j=1}^{n}X_{ij}^{\alpha_{ij}}\geq 1.$$

Set up the Lagrangian

$$\mathcal{L}=L_i+\sum_{j=1}^{n}P_jX_{ij}$$
$$+\lambda\left(e^{\varepsilon_i}A_i(\alpha_i)\zeta(\alpha_i)L_i^{\sum_{j=1}^n 1-\alpha_{ij}}\prod_{j=1}^{n}X_{ij}^{\alpha_{ij}}-1\right).$$

The first-order conditions are

$$\frac{\partial\mathcal{L}}{\partial L_i}=1+\lambda\left(1-\sum_{j=1}^{n}\alpha_{ij}\right)e^{\varepsilon_i}A_i(\alpha_i)\zeta(\alpha_i)L_i^{-\sum_{j=1}^n\alpha_{ij}}\prod_{j=1}^{n}X_{ij}^{\alpha_{ij}}=0, \tag{3}$$

$$\frac{\partial\mathcal{L}}{\partial X_{ij}}=P_j+\lambda\alpha_{ij}e^{\varepsilon_i}A_i(\alpha_i)\zeta(\alpha_i)L_i^{1-\sum_{j=1}^n\alpha_{ij}}\left(\prod_{j'\neq j}X_{ij'}^{\alpha_{ij'}}\right)X_{ij}^{\alpha_{ij}-1}=0 \tag{4}$$

for any $j = 1, \ldots, n$, and

$$\frac{\partial \mathcal{L}}{\partial \lambda} = e^{\varepsilon_i} A_i(\alpha_i) \zeta(\alpha_i) L_i^{1-\sum_{j=1}^n \alpha_{ij}} \prod_{j=1}^n X_{ij}^{\alpha_{ij}} - 1 = 0. \tag{5}$$

Rearranging (3) and (4) and taking the ratio of these two yields

$$\frac{1}{P_j} = \left(\frac{1 - \sum_{j'=1}^n \alpha_{ij'}}{\alpha_{ij}} \right) \cdot \frac{X_{ij}}{L_i}.$$

Solving this for X_{ij},

$$X_{ij} = \frac{1}{P_j} \left(\frac{\alpha_{ij}}{1 - \sum_{j'=1}^n \alpha_{ij'}} \right) L_i \tag{6}$$

for any j. Substituting this into (5) yields

$$e^{\varepsilon_i} A_i(\alpha_i) \zeta(\alpha_i) L_i^{1-\sum_{j=1}^n \alpha_{ij}} \cdot \prod_{j=1}^n \left[\frac{1}{P_j} \left(\frac{\alpha_{ij}}{1 - \sum_{j'=1}^n \alpha_{ij'}} \right) L_i \right]^{\alpha_{ij}} = 1.$$

Rearranging this,

$$e^{\varepsilon_i} A_i(\alpha_i) \zeta(\alpha_i) L_i \cdot \left(\prod_{j=1}^n \left(\frac{1}{P_j} \right)^{\alpha_{ij}} \right) \cdot \left(\prod_{j=1}^n \alpha_{ij}^{\alpha_{ij}} \right) \cdot \left(\prod_{j=1}^n \left(\frac{1}{1 - \sum_{j'=1}^n \alpha_{ij'}} \right)^{\alpha_{ij}} \right) = 1.$$

Using the definition of $\zeta(\alpha_i)$ (See p.8 of `7_production_network.pdf`.),

$$e^{\varepsilon_i} A_i(\alpha_i) L_i \cdot \left(\prod_{j=1}^n \left(\frac{1}{P_j} \right)^{\alpha_{ij}} \right) \cdot \left(1 - \sum_{j=1}^n \alpha_{ij} \right)^{-1} = 1.$$

Solving this for L_i,

$$L_i = \left(1 - \sum_{j'=1}^n \alpha_{ij'} \right) \cdot \prod_{j=1}^n P_j^{\alpha_{ij}} \cdot \frac{1}{e^{\varepsilon_i} A_i(\alpha_i)}. \tag{7}$$

Substituting this into (6) yields

$$X_j = \frac{\alpha_{ij}}{P_j} \cdot \left(\prod_{j'=1}^n P_{j'}^{\alpha_{ij'}} \right) \cdot \frac{1}{e^{\varepsilon_i} A_i(\alpha_i)}. \tag{8}$$

These are the cost-minimizing labor/intermediate inputs. Plugging (7) & (8) into the objective function

(2):

$$\left(1-\sum_{j'=1}^{n}\alpha_{ij'}\right)\cdot\prod_{j=1}^{n}P_j^{\alpha_{ij}}\cdot\frac{1}{e^{\varepsilon_i}A_i(\alpha_i)}+\sum_{j'=1}^{n}P_{j'}\cdot\frac{\alpha_{ij'}}{P_j'}\cdot\left(\prod_{j=1}^{n}P_j^{\alpha_{ij}}\right)\cdot\frac{1}{e^{\varepsilon_i}A_i(\alpha_i)}$$
$$=\frac{\prod_{j=1}^{n}P_j^{\alpha_{ij}}}{e^{\varepsilon_i}A_i(\alpha_i)}.$$

2. To compute the Hessian, first we compute the first derivatives.

$$\frac{\partial a_2}{\partial \alpha_{21}}=-2(\alpha_{21}-0.4)-2(\alpha_{21}-\alpha_{22}-0.1),$$

$$\frac{\partial a_2}{\partial \alpha_{22}}=-2(\alpha_{22}-0.4)+2(\alpha_{21}-\alpha_{22}-0.1).$$

The second derivatives are

$$\frac{\partial^2 a_2}{\partial \alpha_{21}^2}=-4,$$
$$\frac{\partial^2 a_2}{\partial \alpha_{22}^2}=-4,$$

and because of Young's theorem, we have

$$\frac{\partial^2 a_2}{\partial \alpha_{21}\partial \alpha_{22}}=\frac{\partial^2 a_2}{\partial \alpha_{22}\partial \alpha_{21}}=-2.$$

Therefore, the Hessian is

$$\begin{bmatrix}-4 & 2\\ 2 & -4\end{bmatrix}.$$

Its inverse matrix is

$$\frac{1}{16-4}\begin{bmatrix}-4 & -2\\ -4 & -2\end{bmatrix}=\begin{bmatrix}-1/3 & -1/6\\ -1/6 & -1/3\end{bmatrix}.$$

The off-diagonal elements are -1/6. Inputs 1 and 2 are complements for sector 2.